MST 204.6

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods.

mathematical models and methods

unit 6
Differential equations II

The Open University

Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

Unit 6
Differential equations II

Prepared for the Course Team
by Bob Tunnicliffe

The Open University

The Open University, Walton Hall, Milton Keynes.

First published 1982. Reprinted 1983, 1985, 1989, 1992, 1994, 1996.

ISBN 0 335 14035 1

Printed and bound in the United Kingdom by Staples Printers Rochester Limited, Neptune Close, Medway City Estate, Frindsbury, Rochester, Kent ME2 4LT.

This text forms part of the correspondence element of an Open University Second Level Course.

For general availability of supporting material referred to in this text, please write to: Open University Educational Enterprises Limited, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, Great Britain.

Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Milton Keynes, MK7 6AB.

1.7

Contents

Introduction

In *Unit 4* you met *Newton's second law* and *the law of terrestrial gravitation*. If we consider an object falling near the earth's surface, and ignore forces other than gravity acting on the object, then these laws lead to a model of its motion in which the object's *acceleration is constant*. If the height of such an object above ground level is x at time t then the mathematical statement of this model is

$$\frac{dv}{dt} = -g$$

where $v = \dfrac{dx}{dt}$ is the velocity and g is the gravitational acceleration. This equation can also be written

$$\frac{d^2x}{dt^2} = -g \tag{1}$$

where $\dfrac{d^2x}{dt^2}$ is an abbreviation for the second derivative $\dfrac{d}{dt}\left(\dfrac{dx}{dt}\right)$ of x with respect to t. Because Equation (1) contains a second-order derivative it is called a **second-order** differential equation, and as such is an example of the sort of equations that will be discussed in this unit.

In *Unit 4* we saw that Equation (1) can be solved rather simply, by direct integration. We have

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = -g.$$

Integrating with respect to t gives

$$\begin{aligned}\frac{dx}{dt} &= -\int g\,dt + A\\ &= -gt + A\end{aligned}$$

where A is a constant. Integrating a second time gives

$$x = -\tfrac{1}{2}gt^2 + At + B \tag{2}$$

where B is another constant. Equation (2) is the *general* solution of the second-order differential Equation (1). The constants A and B in the general solution may take any value—they are *arbitrary constants*.

This example illustrates some general features of second-order differential equations. The appearance of *two* arbitrary constants in the general solution is typical of *second*-order differential equations. In *Unit 2* you saw that *one* condition was needed to pick out a *particular* solution of a *first*-order differential equation. For a *second*-order equation, *two* conditions are required to pick out a particular solution. These conditions set values on the two arbitrary constants. In the example, we can find values for the constants A and B if we know the position x and velocity $\dfrac{dx}{dt}$ at time $t = 0$. Equation (2) can then be used to predict the position of the object at all future times t.

The example also illustrates very well why second-order differential equations are important. They arise commonly in modelling. In particular, they arise frequently in dynamics, because dynamical models often involve the use of Newton's second law. Mathematical statements which are derived using this law involve acceleration, the second time-derivative of position. However, our concern in this unit will be with the mathematics of solving second-order differential equations. Elsewhere in the course you will see many examples of their use in modelling.

We have seen that the general solution of Equation (1) can be found by direct integration. However, this method only works for equations of the form

$$\frac{d^2y}{dt^2} = f(t),$$

(where the integrals of $f(t)$ and $\int f(t)dt$ can be found explicitly). Most of this unit is devoted to describing a method of finding analytical solutions of a wider class of second-order differential equations. The technique does not depend on integration (it is easier!). Rather it is a matter of recognizing that the equation is of a certain type, for which a set routine then provides a solution.

By 'an analytical solution' we mean a solution which can be written in terms of a formula.

The second-order differential equations with which we are mainly concerned are linear constant-coefficient equations. (I will define these terms shortly.) An example is

$$2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = 4 \sin 2t. \qquad (3)$$

Equations of this type have the advantage of being relatively straightforward to solve analytically.

Equations which are not of this special type may not be so cooperative. In such cases numerical methods may have to be used. The basis of one such method is described in Section 5.

Terminology and notation

You will need to become familiar with certain jargon associated with second-order differential equations. (You may recognize some of it from *Units 1* and *2*.) We shall be exclusively concerned with linear equations.

> A second-order differential equation is **linear** if it can be brought to the form
>
> $$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = f(x) \qquad (4)$$
>
> where p, q, r and f are known, continuous, functions, and p is not the zero function.

If we allow p to be the zero function then Equation (4) includes the first-order linear differential equations considered in *Unit 2*. In this unit we wish to confine our attention to second-order equations and so we impose the condition that p is not the zero function.

I shall make a habit of writing linear equations in the form of Equation (4), with all the terms involving the dependent variable (in this case y) on the left, so that the right-hand side is a function of the independent variable (in this case x) only.

Obviously, letters other than y and x may also be used for the variables.

The equations we will be particularly concerned with are **constant-coefficient** equations, that is, equations for which p, q and r are constant functions. Equation (3) above is an example of a linear, constant coefficient, equation.

We concern ourselves here only with linear constant-coefficient equations because, as I remarked earlier, these are relatively easy to solve. However, even when the coefficients are not constant the idea of a *linear* differential equation is still important because there are a number of general theorems about differential equations which apply to linear equations.

There is one last piece of terminology which we shall need when we study methods of solution. An equation which can be brought to the form

$$p(x)\,\frac{d^2y}{dx^2} + q(x)\,\frac{dy}{dx} + r(x)\,y = 0$$

(that is, Equation (4) with the zero function on the right-hand side) is a **homogeneous** linear differential equation. If the right-hand side of Equation (4) is non-zero, then we have an **inhomogeneous** equation.

Exercise 1

(i) Write down the order of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} = x^2$

(b) $\dfrac{dy}{dx} + 3y = 0$

(c) $\left(\dfrac{d^2y}{dx^2}\right)^2 + \dfrac{dy}{dx} + 3 = 0$

(d) $2\dfrac{d^2y}{dx^2} + xy = 3\dfrac{dy}{dx}$

(ii) Which of the equations in (i) are linear?

(iii) Which of the following second-order linear differential equations are constant-coefficient equations?

(a) $x\dfrac{d^2y}{dx^2} + x^2y = 0$

(b) $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 4y = \sin x$

(c) $\dfrac{d^2y}{dx^2} + 4y = 0$

(d) $2\dfrac{d^2y}{dx^2} + \cos x\dfrac{dy}{dx} + 3xy = e^x$

(iv) Which of the equations in (iii) are homogeneous?

[*Solutions on p. 49*]

Comment on theorems and proofs

Certain important results about differential equations are stated in this unit as theorems. Since the objective of this unit is to teach you how to solve certain differential equations, proofs of theorems are not given. Theorems are stated where they give important information about the solutions of differential equations, and I will try to illustrate their importance.

Many theorems require the condition that all the functions involved are twice differentiable throughout a suitable interval, and that this second derivative is continuous. I shall assume throughout the unit that all functions mentioned satisfy these conditions, without stating this explicitly in the theorems.

Providing that you can solve the differential equations, understanding the theorems is only a secondary objective of this unit. No knowledge of the proofs is required. All the proofs omitted may be found in text books such as *An Introduction to Linear Analysis* by Kreider, Kuller, Ostberg and Perkins (Addison-Wesley 1966), and many also in *M203: Introduction to Pure Mathematics.*

Study guide

The first four sections of this unit are concerned with linear constant-coefficient equations. These are equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, and c are constants (and $a \neq 0$). In Section 1 we examine the *homogeneous* case, where f is the zero function. The *inhomogeneous* case, where f is some non-zero function, is considered in Section 2.

Section 3 is concerned with finding particular solutions that fit given conditions (such as $y = 2$ and $\dfrac{dy}{dx} = 1$ when $x = 0$). There is an audio-tape associated with this section. Section 4 is the television section, and examines the graphs of some of the functions that arise as solutions of linear constant-coefficient equations. Section 5 concerns a numerical method for second-order differential equations. Section 6 contains additional exercises.

The television section makes some reference to the material in Sections 1 to 3 and so it would help to have studied these sections before watching the programme. However, if you have not finished the first three sections by the time the television programme is due to be broadcast, make sure you have read the preliminary notes at the beginning of Section 4 before watching the programme. The audio-tape cannot be studied until you have completed Sections 1 and 2. Although there are five sections of expository text you should not find the unit excessively long since Sections 3, 4 and 5 are all fairly short.

1 Homogeneous equations

1.0 Introduction

In this section we are concerned with the solution of equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (1)$$

The general homogeneous constant-coefficient second-order differential equation.

where a, b and c are constants, and a is not zero. I shall first describe (in Subsection 1.1) the method whereby the general solution of an equation of this form can be found. Afterwards, in Subsection 1.2, I shall explain why the method works.

1.1 The method of solution

The auxiliary equation

To indicate the idea of the method we consider the linear *first*-order differential equation

$$\frac{dy}{dx} - \lambda y = 0.$$

One solution of this equation is $y = e^{\lambda x}$. This suggests that we might try the same exponential form for the solution of the second-order Equation (1).

To try $y = e^{\lambda x}$ in Equation (1) we first calculate

$$\frac{dy}{dx} = \lambda e^{\lambda x}$$

and

$$\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x},$$

so that

$$\begin{aligned} a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} \\ &= (a\lambda^2 + b\lambda + c)e^{\lambda x}. \end{aligned}$$

Thus, $y = e^{\lambda x}$ is a solution of Equation (1) provided that

$$a\lambda^2 + b\lambda + c = 0.$$

This equation plays such an important role in solving the differential equation that we give it a special name:

> **Definition 1**
> The quadratic equation
> $$a\lambda^2 + b\lambda + c = 0$$
> is called the **auxiliary equation** of the Differential equation (1).

Example 1

The auxiliary equation of the differential equation

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$$

is

$$3\lambda^2 - 2\lambda + 4 = 0.$$

Solving the differential equation

You should recall that the solution of a quadratic equation with real coefficients (such as the auxiliary equation) may be divided into three cases:

(i) The equation has two distinct, real, solutions.

(ii) The equation has one real solution (two equal roots).

(iii) The equation has two conjugate complex solutions.

These three cases for the solution of the auxiliary equation correspond to three cases for the solution of the Differential Equation (1). In the procedure below I describe how to write down the solution of Equation (1) in each case. Later in this section (Subsection 1.2) I shall explain why the solutions take the form they do. For now, though, I am only concerned with the technique.

Procedure 1.1

To solve the homogeneous differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where a, b and c are constants, and $a \neq 0$.

1. Write down the auxiliary equation

$$a\lambda^2 + b\lambda + c = 0$$

and solve it.

2. The solution of the differential equation takes a different form in each of three cases, depending on the type of solution of the auxiliary equation.

(i) If the auxiliary equation has two, distinct, real solutions $\lambda = \lambda_1$ and $\lambda = \lambda_2$, then the general solution of the differential equation is

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

(ii) If the auxiliary equation has equal roots, so that it has just one solution $\lambda = \alpha$, then the general solution of the differential equation is

$$y = (A + Bx)e^{\alpha x}$$

(iii) If the auxiliary equation has complex roots, $\lambda = \alpha + i\beta$ and $\lambda = \alpha - i\beta$, then the general solution of the differential equation is

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x).$$

In each case A and B are arbitrary constants.

The remainder of this subsection consists of worked examples illustrating the procedure above. You may wish to read through these, or you may prefer to try the examples yourself first, and refer to their solutions only if you get stuck.

Example 2

Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

Method

The auxiliary equation is

$$\lambda^2 - 3\lambda + 2 = 0.$$

Its solutions are $\lambda = 1$, $\lambda = 2$. This is Case (i) of Procedure 1.1 and so the general solution of the differential equation is

$$y = Ae^{x} + Be^{2x}$$

where A and B are arbitrary constants.

Example 3

Solve the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

Method

The auxiliary equation is

$$\lambda^2 + 4 = 0.$$

The solutions of this are $\lambda = 2i$, $\lambda = -2i$. This is Case (iii) of the Procedure 1.1, with $\alpha = 0$, $\beta = 2$. Hence the general solution of the differential equation is

$$y = A\cos 2x + B\sin 2x$$

where A and B are arbitrary constants.

You may wish to check the solutions obtained in these examples by differentiation and substitution.

Example 4

Solve the differential equation

$$4\frac{d^2u}{dt^2} + 4\frac{du}{dt} + u = 0.$$

Method

The letters used for the variables are different here, but this makes no difference to the method. The auxiliary equation is

$$4\lambda^2 + 4\lambda + 1 = 0$$

The left-hand side is the perfect square $(2\lambda + 1)^2$ and so the only solution is $\lambda = -\frac{1}{2}$. This is Case (ii) of Procedure 1.1, so the general solution of the differential equation is

$$u = (A + Bt)e^{-\frac{1}{2}t},$$

where A and B are arbitrary constants.

Example 5

Solve the differential equation

$$2\frac{d^2z}{d\theta^2} - 3\frac{dz}{d\theta} + 2z = 0.$$

Method

The auxiliary equation is

$$2\lambda^2 - 3\lambda + 2 = 0.$$

Using the formula for solving the quadratic equation gives

$$\lambda = \frac{3 \pm \sqrt{9-16}}{4} = \frac{3}{4} \pm i\frac{\sqrt{7}}{4}.$$

This is Case (iii) of Procedure 1.1, with $\alpha = \frac{3}{4}$, $\beta = \frac{\sqrt{7}}{4}$. So the general solution of the differential equation is

$$z = \exp\left(\frac{3}{4}\theta\right)\left(A\cos\left(\frac{\sqrt{7}}{4}\theta\right) + B\sin\left(\frac{\sqrt{7}}{4}\theta\right)\right),$$

where A and B are arbitrary constants.

Example 6 (*Where $\lambda = 0$ is a solution of the auxiliary equation.*)

Solve

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$$

Method

The solutions of the auxiliary equation $\lambda^2 + 3\lambda = 0$ are $\lambda = 0$, $\lambda = -3$. This is Case (i) of Procedure 1.1. The only difficulty lies in realizing that $e^{0x} = e^0 = 1$, so that the general solution is

$$y = A + Be^{-3x},$$

where A and B are arbitrary constants.

Exercise 1 (*Practice in solving homogeneous differential equations.*)

Solve each of the differential equations below.

(i) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 5y = 0$

(ii) $\dfrac{d^2\theta}{dt^2} + 9\theta = 0$

(iii) $\dfrac{d^2z}{du^2} - 4z = 0$

(iv) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

(v) $\dfrac{d^2u}{dt^2} + 4\dfrac{du}{dt} + 8u = 0$

(vi) $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} = 0.$

[*Solution on p. 49*]

Sometimes we need to solve equations which have coefficients expressed in terms of unspecified constants rather than numbers. This does not affect the method of solution.

Example 7

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4\omega\frac{dx}{dt} + \omega^2 x = 0$$

where ω is a constant.

Method

The auxiliary equation is

$$\lambda^2 + 4\omega\lambda + \omega^2 = 0.$$

The values of λ satisfying this are

$$\lambda = \frac{-4\omega \pm \sqrt{16\omega^2 - 4\omega^2}}{2} = (-2 \pm \sqrt{3})\omega.$$

We have distinct, real, roots (Case (i) of Procedure 1.1). The general solution of the differential equation is therefore

$$x = Ae^{(-2+\sqrt{3})\omega t} + Be^{(-2-\sqrt{3})\omega t},$$

where A and B are arbitrary constants.

Exercise 2

Find the general solution of each of the differential equations below.

(i) $\dfrac{d^2y}{dx^2} + 2\omega\dfrac{dy}{dx} + \omega^2 y = 0$

(ii) $5\dfrac{d^2x}{dt^2} + 6\omega\dfrac{dx}{dt} + 5\omega^2 x = 0$

where ω is a constant.

[*Solution on p. 49*]

1.2 Why the method works

We now look at why the expressions given in Procedure 1.1 for the general solution of a homogeneous equation (Equation (1)) are correct. Before looking at the specific expressions given for the solutions, I will discuss some ideas relevant to the solution of *linear* differential equations in general.

> **Definition 2**
>
> If u_1 and u_2 are functions, then a **linear combination** of u_1 and u_2 is any function u of the form
>
> $$u(x) = c_1u_1(x) + c_2u_2(x)$$
>
> where c_1 and c_2 are *constants*.

Example 8

Suppose $u_1(x) = x^2$, $u_2(x) = \sin x$. The functions

$$u(x) = 3x^2 + 2\sin x,$$
$$u(x) = 2x^2 - 4\sin x,$$

are linear combinations of u_1 and u_2. On the other hand the functions

$$u(x) = x^2 \sin x,$$
$$u(x) = x^2 + \sin 2x,$$
$$u(x) = 2\sin x + x^4,$$

are *not* linear combinations of u_1 and u_2.

> **Theorem 1**
>
> Suppose that the functions u_1 and u_2 are solutions of a *homogeneous* linear differential equation. Then any linear combination of u_1 and u_2 is also a solution of this differential equation.

This theorem is fairly easy to prove.

The next example illustrates how this theorem can help us validate the method in Procedure 1.1.

Example 9

Consider the differential equation

$$u''(x) - 3u'(x) + 2u(x) = 0 \tag{2}$$

Remember that $u'(x)$ means $\dfrac{du(x)}{dx}$

Each of the functions $u_1(x) = e^x$ and $u_2(x) = e^{2x}$ is a solution of this differential equation. (This follows from the fact that $\lambda = 1$ and $\lambda = 2$ are solutions of the auxiliary equation $\lambda^2 - 3\lambda + 2 = 0$.) We now know, from Theorem 1, that any linear combination of u_1 and u_2 is also a solution of Equation (2). That is, any function

$$u(x) = Au_1(x) + Bu_2(x) = Ae^x + Be^{2x},$$

where A and B are constants, also satisfies Equation (2). For example, one solution is $2e^x + 3e^{2x}$.

In each of the three cases described in Procedure 1.1, the formula given for the solution of Equation (1) is a linear combination of two functions.

Case (i): $Ae^{\lambda_1 x} + Be^{\lambda_2 x}$.

Case (ii): $Ae^{\alpha x} + Bxe^{\alpha x}$.

Case (iii): $Ae^{\alpha x}\cos \beta x + Be^{\alpha x}\sin \beta x$.

Theorem 1 shows that so long as we know that the individual functions ($e^{\lambda_1 x}$, $e^{\lambda_2 x}$ in Case (i); $e^{\alpha x}$ and $xe^{\alpha x}$ in Case (ii); $e^{\alpha x}\sin \beta x$ and $e^{\alpha x}\cos \beta x$ in Case (iii)) are solutions of Equation (1), then these linear combinations must also be solutions. To check that the individual functions are in fact solutions is not too complicated. Let us look at this now.

Checking the solutions

The general homogeneous constant-coefficient linear second-order differential equation (1) may alternatively be written as

$$au''(x) + bu'(x) + cu(x) = 0 \tag{3}$$

(a notation which emphasizes that the solutions $u(x)$ are functions). Now we saw in Subsection 1.1 that $u(x) = e^{\lambda x}$ is a solution of Equation (3) so long as λ satisfies the auxiliary equation

$$a\lambda^2 + b\lambda + c = 0. \tag{4}$$

Let us now check that the functions specified in Procedure 1.1 are solutions of Equation (3) in the various cases.

Case (i)
In this case $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are real distinct roots of Equation (4). Thus the functions $u_1(x) = e^{\lambda_1 x}$ and $u_2(x) = e^{\lambda_2 x}$ are already known to be solutions of Equation (3).

Case (ii)
In this case $\lambda = \alpha$ is the only solution of Equation (4). Thus $u_1(x) = e^{\alpha x}$ is known to satisfy Equation (3). The expression given in Procedure 1.1 for the general solution of Equation (3) also contains the function $u_2(x) = xe^{\alpha x}$. We must check that this is also a solution. We have

$$u_2(x) = xe^{\alpha x}$$
$$u_2'(x) = e^{\alpha x} + \alpha xe^{\alpha x} = (1 + \alpha x)e^{\alpha x}$$
$$u_2''(x) = \alpha e^{\alpha x} + (1 + \alpha x)\alpha e^{\alpha x} = (2\alpha + \alpha^2 x)e^{\alpha x}.$$

Hence

$$\begin{aligned} au_2''(x) + bu_2'(x) + cu_2(x) &= e^{\alpha x}(a(2\alpha + \alpha^2 x) + b(1 + \alpha x) + cx) \\ &= e^{\alpha x}((2a\alpha + b) + (a\alpha^2 + b\alpha + c)x). \end{aligned} \tag{5}$$

Now α is a solution of Equation (4); so $a\alpha^2 + b\alpha + c = 0$.

Also, we know that the auxiliary equation has coincident roots, so that $b^2 - 4ac = 0$. Hence its solution is $\alpha = -\dfrac{b}{2a}$, and so $2a\alpha + b = 0$.

Thus the right-hand side of Equation (5) above is indeed zero, and so u_2 is a solution of Equation (3).

Case (iii)
We know that $\lambda = \alpha + i\beta$ and $\lambda = \alpha - i\beta$ are solutions of Equation (4) and so the complex functions

$$u_1(x) = e^{(\alpha + i\beta)x}$$

and

$$u_2(x) = e^{(\alpha - i\beta)x}$$

are solutions of Equation (3). In this course we will normally restrict our attention to *real* solutions of differential equations, and so the complex solutions u_1 and u_2 are not suitable for our purposes. However, using Euler's formula (see *Unit 5*, Section 4) we can write:

$$e^{\alpha x}\cos\beta x = \frac{1}{2}u_1(x) + \frac{1}{2}u_2(x)$$

and

$$e^{\alpha x}\sin\beta x = \frac{1}{2i}u_1(x) - \frac{1}{2i}u_2(x).$$

Hence, by Theorem 1, the (real) functions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are also solutions of Equation (3). These are the functions appearing in the formula in Procedure 1.1.

To make this argument precise we should define $\dfrac{d}{dt}$ when z is a complex variable. If $w(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real functions of t, then we define
$$\frac{dw(t)}{dt} = \frac{du(t)}{dt} + i\frac{dv(t)}{dt}.$$
It is then possible to show that many of the usual rules of differentiation still apply. In particular
$$\frac{de^{i\beta t}}{dt} = i\beta e^{i\beta t}$$
Furthermore, having defined the derivative of a complex variable, it becomes meaningful to talk about complex solutions of differential equations.

We shall normally invoke Theorem 1 in the context of *real* solutions, but the theorem is true for *complex* solutions as well.

The general solution

So far we have seen why the functions given by Procedure 1.1 are solutions of Equation (3). To show that the procedure gives the *general* solution of Equation (3) we must show that there are no other (real) solutions. This is the consequence of another general theorem.

The basic idea of the theorem is that once we have found two 'genuinely different' solutions of a linear second-order homogeneous differential equation, then the general solution is given by an arbitrary linear combination of them. To state the Theorem we need the following definition of what 'genuinely different' means.

Definition 3

Two functions u_1 and u_2 having the same domain are **linearly independent** if neither function is a constant multiple of the other

The function u_1 is a constant multiple of u_2 if both functions have the same domain and if there is a constant λ such that

$$u_1(x) = \lambda u_2(x)$$

for all x in their common domain

Example 10

(i) If $u_1(x) = e^x$ and $u_2(x) = e^{2x}$ then u_1 and u_2 are linearly independent. For there is no constant λ such that $e^{2x} = \lambda e^x$ (or $e^x = \lambda e^{2x}$) for all real numbers x.

(ii) If $u_1(x) = \sin x$ and $u_2(x) = 2\sin x$ then u_1 and u_2 are *not* linearly independent since $u_2(x) = 2u_1(x)$.

Theorem 2

Suppose that D is an interval of real numbers and that the functions u_1 and u_2 both have domain D and are linearly independent solutions of the differential equation

$$p(x)u''(x) + q(x)u'(x) + r(x)u(x) = 0 \quad \text{(for all } x \text{ in } D) \qquad (6)$$

Suppose also that for every x in D, $p(x)$ is not equal to zero.

Then the general solution of Equation (6) is $Au_1 + Bu_2$ where A and B are arbitrary constants.

A definition of **interval** is given in the study comments for Unit 6 in Section 5 of the *Preparatory Booklet*

I have stated the theorem in a form applicable to general second-order homogeneous linear differential equations. Notice that a second-order equation with constant coefficients inevitably satisfies the condition $p(x) \neq 0$ for all x. (If $p(x) = a = 0$, Equation (6) is not a second-order equation!)

My main aim in this section has been to explain why the method given in Subsection 1.1 for constant coefficient equations works. However, the theorems are informative for non-constant-coefficient equations also.

Example 11

The general solution of

$$x^2u''(x) + xu'(x) - u(x) = 0 \qquad (x > 0) \qquad (7)$$

is

$$u(x) = Ax + \frac{B}{x} \qquad (x > 0).$$

Explanation

Let $u_1(x) = x$ and $u_2(x) = \frac{1}{x}$ be functions which have the same domain $(x > 0)$.

We can show that these functions are solutions of Equation (7) by direct substitution. (I leave you to do this.)

Now u_1 and u_2 are linearly independent functions. Also, Equation (7) is of the form described in Theorem 2 (for in this case D is the interval $(x > 0)$ and so $p(x) = x^2$ is non-zero for all x in D.

Hence, by Theorem 2, the general solution of Equation (7) is

$$u(x) = \frac{A}{x} + Bx \qquad (x > 0),$$

where A and B are arbitrary constants.

Exercise 3

Find the general solution of the differential equation

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \qquad (x > 0)$$

Hint: first look for solutions of the form $y = x^n$.

[*Solution on p. 49*]

Summary of Section 1

The general solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

depends on the nature of the solution of its **auxiliary equation**

$$a\lambda^2 + b\lambda + c = 0.$$

The following table summarizes the method:

Type of solution of auxiliary equation	General solution of differential equation. (In each case A and B are arbitrary constants.)
Two real roots: $\lambda = \lambda_1$ and $\lambda = \lambda_2$	$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
Equal roots (one real solution): $\lambda = \alpha$	$y = (A + Bx)e^{\alpha x}$
Complex roots: $\lambda = \alpha + i\beta$ and $\lambda = \alpha - i\beta$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

This method may be justified by:

1. verifying that the various functions ($e^{\lambda_1 x}$, $e^{\lambda_2 x}$, $e^{\alpha x}$, $xe^{\alpha x}$, $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$) are indeed solutions in each case;
2. referring to Theorems 1 and 2 stated in the text.

The concepts of **linear combination** (Definition 2) and **linear independence** (Definition 3) for two functions were defined in the text.

End of section exercises

Exercise 4

(i) What is the auxiliary equation of the differential equation

$$3\frac{dy}{dx} - y - 2\frac{d^2y}{dx^2} = 0.$$

(ii) Solve this auxiliary equation.

(iii) Write down the general solution of the differential equation.

[*Solution on p. 49*]

Exercise 5

Find the general solution of each of the differential equations below.

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$

(ii) $\dfrac{d^2y}{dx^2} - 16y = 0$

(iii) $\dfrac{d^2y}{dx^2} + 4y = 4\dfrac{dy}{dx}$

(iv) $\dfrac{d^2\theta}{dt^2} = 0.$

[*Solution on p. 50*]

Exercise 6

For which values of the constant k does the differential equation below have a general solution that involves sines and cosines?

$$\frac{d^2y}{dx^2} + 4k\frac{dy}{dx} + 4y = 0.$$

[*Solution on p. 50*]

Exercise 7

Given that $y = x$ and $y = x^2$ are solutions of

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \qquad \text{(for all real } x\text{)}$$

is it legitimate to use Theorem 2 to write down the general solution of this differential equation?

[*Solution on p. 50*]

2 Inhomogeneous equations

2.0 Introduction

In this section we study the solution of equations of the form

$$au''(x) + bu'(x) + cu(x) = f(x) \tag{1}$$

where a, b and c are constants. These differ from the homogeneous equations studied in Section 1 in that a function $f(x)$ has been introduced on the right-hand side.

2.1 The method of solution: 'particular solution plus complementary function'

In essence, the method of solution is similar to the method used in *Unit 1* for solving inhomogeneous recurrence relations. The method depends on our being able to find a *particular* solution of Equation (1). Adding any such particular solution to the general solution of the *homogeneous* equation

$$au''(x) + bu'(x) + cu(x) = 0$$

(obtained by replacing $f(x)$ in Equation (1) by zero) gives the *general* solution of Equation (1).

Before examining the theoretical justification for this method I will work through an example that illustrates how it works in practice.

Example 1

Find the general solution of the equation

$$u''(x) - 3u'(x) + 2u(x) = 6. \tag{2}$$

Method

Step 1 We first solve the homogeneous equation obtained by replacing the right-hand side of Equation (2) by zero. That is, we find the general solution of

$$u''(x) - 3u'(x) + 2u(x) = 0.$$

Apart from a change of notation this equation is the same as the equation in Example 2 of Section 1. There we obtained the solution $u = u_h$, where

$$u_h(x) = Ae^x + Be^{2x}.$$

(A and B are arbitrary constants.)

Step 2 Next we find any one solution of the Inhomogeneous equation (2). This is not always easy, but in this case it is quite easily seen that the constant function

$$u_p(x) = 3$$

is a solution. (For if $u_p(x) = 3$, then $u_p'(x) = 0$ and $u_p''(x) = 0$. Thus $u_p''(x) - 3u_p'(x) + 2u_p(x) = 6$, as required.)

Step 3 Finally we add the particular solution found in Step 2 to the general solution of the homogeneous equation found in Step 1. The result $u_h(x) + u_p(x)$ is the general solution of the original Inhomogeneous equation (2). Thus

$$u(x) = Ae^x + Be^{2x} + 3$$

is the general solution of Equation (2).

The theoretical justification of this method relies on a theorem (Theorem 1 below). Although we are concerned with constant coefficient second-order equations in this unit, the method works much more generally, and so I will state the theorem for general second-order linear equations. To do so I will need the following definition.

Definition 1

Let

$$p(x)u''(x) + q(x)u'(x) + r(x)u(x) = f(x) \tag{3}$$

be any inhomogeneous linear second-order differential equation. Then its **associated homogeneous equation** is

$$p(x)u''(x) + q(x)u'(x) + r(x)u(x) = 0.$$

Theorem 1

Suppose that u_p is any solution of Equation (3) and that u_h is the general solution of the associated homogeneous equation. Then $u_p + u_h$ is the general solution of Equation (3).

We often need to refer to 'the general solution of the associated homogeneous equation' and so it is convenient to give it a special name:

Definition 2

The general solution of the associated homogeneous equation is known as the **complementary function**.

With this definition we can make a succinct statement of the method for solving inhomogeneous equations of the form of Equation (1):

Procedure 2.1

To solve the inhomogeneous differential equation

$$au''(x) + bu'(x) + cu(x) = f(x).$$

1. Find the complementary function using Procedure 1.1.
2. Find a particular solution.
3. General solution
   = Particular solution + Complementary function.

Example 2

Find the general solution of the equation

$$u''(x) + 9u(x) = 9x. \tag{4}$$

Method

The associated homogeneous equation is

$$u''(x) + 9u(x) = 0$$

which has the general solution $u = u_h$, where

$$u_h(x) = A\cos 3x + B\sin 3x.$$

This is the complementary function of Equation (4).

A particular solution of Equation (4) is the function

$$u_p(x) = x.$$

(For, if $u_p(x) = x$, then $u_p'(x) = 1$, so $u_p''(x) = 0$. Thus $u_p''(x) + 9u_p(x) = 9x$, as required.)

The general solution of Equation (4) is, therefore,

$$u(x) = x + A\cos 3x + B\sin 3x,$$

where A and B are arbitrary constants.

The finding of the complementary function is a question of applying the method from Section 1. However the finding of a particular solution is another matter. In the two examples above, I have carefully chosen equations for which a particular solution can be found 'by inspection'. This sort of approach is not going to work in general, though. The next subsection looks at the finding of particular solutions in a more systematic way.

Exercise 1

Consider the differential equation

$$\frac{d^2y}{dx^2} + 4y = 8.$$

(i) Write down the associated homogeneous equation and the complementary function.

(ii) Find a particular solution of the form $y = \text{constant}$.

(iii) Write down the general solution of the differential equation.

[*Solution on p. 50*]

There is one point that sometimes causes confusion. An inhomogeneous equation has more than one particular solution (of course!). Does it matter which one we choose to add to the complementary function? It does not, the general solutions obtained may look different, but are actually equivalent.

2.2 Finding a particular solution

I shall not describe a general procedure for finding a particular solution of Equation (1). The form of the particular solution depends on the form of the function $f(x)$ on the right-hand side of the equation. I shall describe the sort of functions that work for certain specific forms of $f(x)$. On the whole, the approach is 'try something similar to $f(x)$'. I will start with a fairly simple case.

***f* is linear** $(f(x) = kx + l)$

The first example I will consider has $f(x) = 4x + 2$. The basis of the method is to try a particular solution which is similar to f. In this case f is linear so I will try an arbitrary linear function $y = mx + n$, and find (if possible) suitable values for the constants m and n for which this is a solution.

Example 3

Find a particular solution of the equation

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4x + 2$$

Solution

We try a solution of the form

$$y = mx + n$$

where m and n are constants to be chosen so that the differential equation is satisfied. As a preliminary to substituting this function into the differential equation we note that $\dfrac{dy}{dx} = m$ and $\dfrac{d^2y}{dx^2} = 0$; hence

$$\begin{aligned} 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y &= 0 - 2m + (mx + n) \\ &= mx + (n - 2m). \end{aligned}$$

For the function $mx + (n - 2m)$ to be equal to the function $4x + 2$ on the right-hand side of the differential equation, we must have

$$mx + (n - 2m) = 4x + 2 \qquad \text{(for all } x\text{)}.$$

This can only be true if

$$m = 4$$

and

$$n - 2m = 2.$$

We can solve these simultaneous equations (by substituting $m = 4$ into $n - 2m = 2$) to give $m = 4$, $n = 10$. Thus

$$y = 4x + 10$$

is a particular solution of the given differential equation.

It is always sensible to check calculations such as these by substituting back into the differential equation as follows.

Check: If $y = 4x + 10$ then

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3 \times 0 - 2 \times 4 + (4x + 10)$$
$$= 4x + 2,$$

as required.

Exercise 2

Find and check particular solutions of the form $y = mx + n$ for each of the differential equations below

(i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 2x + 3.$ (ii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2x$

[*Solution on p. 50*]

(Notice that in part (ii) of the above exercise you still need to look for a solution of the form $y = mx + n$. Although the function $f(x)$ is just a multiple of x in that case, there is no solution of the form $y = mx$.)

***f* an exponential** $(f(x) = ke^{ax})$

In the next example, $f(x) = 3e^{2x}$. 'Try something similar' to $f(x)$ again works. This time we try an exponential, $y = me^{2x}$, and determine the value of the constant m.

We try an exponential with the *same* exponent as f

Example 4

Find a particular solution of

$$\frac{d^2y}{dx^2} + y = 3e^{2x}$$

Solution

We try a solution of the form

$$y = me^{2x}.$$

Then $\dfrac{dy}{dx} = 2me^{2x}$, and $\dfrac{d^2y}{dx^2} = 4me^{2x}$, so $\dfrac{d^2y}{dx^2} + y = 4me^{2x} + me^{2x} = 5me^{2x}$.

Thus $y = me^{2x}$ is a solution of the differential equation if $5me^{2x} = 3e^{2x}$, that is if $m = \dfrac{3}{5}$. Hence

$$y = \frac{3}{5}e^{2x}$$

is a particular solution of the given differential equation.

Exercise 3

Find a particular solution of the equation

$$2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2e^{-x}.$$

[*Solution on p. 50*]

***f* sinusoidal** $(f(t) = k\cos\omega t + l\sin\omega t)$

The most important case so far as applications are concerned arises when the function f is a sine or a cosine or a linear combination of them both. In the next example f is given by $f(t) = 2\sin 3t$. Although there is no cosine term we look for a particular solution of general sinusoidal type $y = m\cos 3t + n\sin 3t$.

Example 5

Find a particular solution of

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 2\sin 3t. \tag{5}$$

Solution

This time we try

$$y = m\cos 3t + n\sin 3t.$$

Then

$$\frac{dy}{dt} = 3n\cos 3t - 3m\sin 3t$$

and

$$\frac{d^2y}{dt^2} = -9m\cos 3t - 9n\sin 3t$$

so

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = (-7m + 3n)\cos 3t + (-3m - 7n)\sin 3t.$$

We require this to equal $2\sin 3t$. So we need

$$-7m + 3n = 0$$
$$-3m - 7n = 2.$$

These equations give $m = -\frac{3}{29}$ and $n = -\frac{7}{29}$. So a particular solution is

$$y = -\frac{3}{29}\cos 3t - \frac{7}{29}\sin 3t.$$

Before asking you to do any examples, I will show you an alternative approach for this example. This uses complex numbers, and is easier so long as you are confident with their use. The method is based on the idea that any sinuosoidal function can be represented by a *phasor* (see *Unit 5*, Subsection 4.3). We look for a particular solution of Equation (5) which is a sinusoidal function (as in the solution above). This time, however, we do the calculations using phasors.

Remember that the phasor of the sinusoidal function

$$y = m\cos\omega t + n\sin\omega t$$

is the complex number z for which

$$y = \operatorname{Re}(ze^{i\omega t})$$

and is given by

$$z = m - in.$$

See Unit 5, Subsection 4.3, Example 4.

With this in mind, we can look for a solution of Equation (5) of the form

$$y = \operatorname{Re}(ze^{3it}) \tag{6}$$

where z is a complex number to be chosen so that the differential equation is satisfied. We have

$$\frac{dy}{dt} = \operatorname{Re}(3ize^{3it})$$

and

$$\frac{d^2y}{dt^2} = \mathrm{Re}\,(-9ze^{3it}).$$

So (6) is a solution of Equation (5) so long as

$$\mathrm{Re}((-9 + 3i + 2)ze^{3it}) = 2\sin 3t$$

Now the phasor of $2\sin 3t$ is $-2i$ so if we write the right-hand side in phasor form we have

$$\mathrm{Re}((-9 + 3i + 2)ze^{3it}) = \mathrm{Re}(-2ie^{3it}).$$

This holds if

$$(-7 + 3i)z = -2i$$

so

$$z = \frac{2i}{7 - 3i} = -\frac{3}{29} + \frac{7}{29}i$$

With this value of z, Equation (6) gives a particular solution of the Differential Equation (5). This particular solution is a sinusoidal function with frequency 3 and phasor $-\frac{3}{29} + \frac{7}{29}i$. It can therefore be written

$$y = -\frac{3}{29}\cos 3t - \frac{7}{29}\sin 3t. \tag{7}$$

This alternative approach may look a little complicated at first but in practice we can abbreviate the calculation considerably. You will find that you can get from Equation (5) to Equation (7) fairly easily by applying the following general procedure:

Procedure 2.2(a)

To find a particular solution of

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = k\cos\omega t + l\sin\omega t.$$

1. Write the right-hand side in phasor form

   $$\mathrm{Re}\,((k - il)e^{i\omega t}).$$

2. Try a solution in the form

   $$y = \mathrm{Re}\,(ze^{i\omega t}).$$

3. Substitute in the differential equation and simplify to obtain

   $$(-\omega^2 a + i\omega b + c)z = k - il,$$

   and solve for z.

4. Then the particular solution is the sinusoid with frequency ω and phasor z. That is, if $z = X + iY$, the solution is

   $$y = X\cos\omega x - Y\sin\omega x.$$

Although the exercises in this unit can be done using either of the methods I have described for Example 5, I recommend that you get used to the complex number method. It is always quicker, and sometimes much quicker (for example, for the sort of problems discussed in Section 4).

Exercise 4

Find a particular solution of the equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} = 9\cos 3t$$

[*Solution on p. 50*]

Exercise 5 (*Optional, additional practice.*)
Find a particular solution of the equation

$$\frac{d^2y}{dt^2} - 4y = 8\cos 2t + 16\sin 2t.$$

[*Solution on p. 51*]

In this subsection we have looked at the problem of finding a particular solution of Equation (1) for certain specific forms of $f(x)$. We have seen that the following procedure usually works.

Procedure 2.2
To find a particular solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x):$$

1. If $f(x) = kx + l$, try

$$y = mx + n.$$

2. If $f(x) = ke^{ax}$, try

$$y = me^{ax}.$$

3. If $f(x) = k\cos\omega x + l\sin\omega x$, try

$$y = m\cos\omega x + n\sin\omega x,$$

or replace the right-hand side by $\text{Re}((k - il)e^{i\omega x})$ and try

$$y = \text{Re}(ze^{i\omega x})$$

as described in Procedure 2.2(a).

I pointedly say that this procedure 'usually' works, for reasons we shall now examine.

2.3 Exceptional cases for particular solutions

Occasionally, the functions suggested in Procedure 2.2 fail to provide a particular solution. Let us see an example of this difficulty, and how it may be resolved.

Example 6
Find a particular solution of the equation

$$\frac{d^2y}{dx^2} - 4y = 2e^{2x}.$$

Solution
According to Procedure 2.2 the obvious guess is $y = me^{2x}$. If we try this,

then $\dfrac{dy}{dx} = 2me^{2x}$ and $\dfrac{d^2y}{dx^2} = 4me^{2x}$, giving

$$\frac{d^2y}{dx^2} - 4y = 4me^{2x} - 4me^{2x} = 0.$$

So there is no value of m which makes $y = me^{2x}$ a solution of the differential equation. What has gone wrong? The trouble is that the solution suggested in Procedure 2.2 contains the function e^{2x} which is a solution of the associated homogeneous equation $\dfrac{d^2y}{dx^2} - 4y = 0$. It follows that me^{2x} is part of the complementary function and so cannot be a solution of the inhomogeneous equation.

When Procedure 2.2 fails in this way we can usually overcome the difficulty by multiplying the function suggested in the procedure by x. In this case we try

$$y = mxe^{2x}.$$

Then

$$\frac{dy}{dx} = me^{2x} + 2mxe^{2x} = m(1 + 2x)e^{2x},$$

and

$$\frac{d^2y}{dx^2} = 2me^{2x} + 2m(1 + 2x)e^{2x} = 4m(1 + x)e^{2x},$$

so that now

$$\frac{d^2y}{dx^2} - 4y = 4m(1 + x)e^{2x} - 4mxe^{2x} = 4me^{2x}.$$

This can be made equal to $2e^{2x}$ by choosing $m = \frac{1}{2}$, which gives the particular solution

$$y = \frac{1}{2}xe^{2x}.$$

In this example we see that the approach described in Procedure 2.2 does not work, but that by 'multiplying by x' we can still find a particular solution.

Let us now look at two other examples where similar difficulties arise, and how they may be resolved.

Example 7

Find a particular solution of the equation

$$\frac{d^2y}{dx^2} + 9y = \sin 3x. \tag{8}$$

Solution

Following Procedure 2.2 we would try $y = m\cos 3x + n\sin 3x$. However, since both $\cos 3x$ and $\sin 3x$ are solutions of the associated homogeneous equation $\frac{d^2y}{dx^2} + 9y = 0$, this would just give zero when put into the left-hand side of Equation (8). To overcome this difficulty we again multiply the function suggested in Procedure 2.2 by x. That is, we try

$$y = x(m\cos 3x + n\sin 3x).$$

We can simplify the calculation by expressing this in the phasor form

$$y = \mathrm{Re}\,(xze^{3ix})$$

where $z = m - in$ is to be chosen so that the differential equation is satisfied. We have

$$\frac{dy}{dx} = \mathrm{Re}\,((1 + 3ix)ze^{3ix})$$

and

$$\frac{d^2y}{dx^2} = \mathrm{Re}\,((6i - 9x)ze^{3ix})$$

so

$$\frac{d^2y}{dx^2} + 9y = \mathrm{Re}\,(6ize^{3ix}).$$

The right-hand side of Equation (8) can be written $\mathrm{Re}(-ie^{3ix})$. If this is to be equal to $\mathrm{Re}(6ize^{3ix})$ we require

$$6iz = -i$$

that is

$$z = -\frac{1}{6}$$

Substituting this into $y = \text{Re}(xze^{3ix})$ gives the particular solution

$$y = -\frac{1}{6}x\cos 3x.$$

Example 8

Find a particular solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2x + 2 \qquad (9)$$

Solution

This time we have a linear function on the right-hand side, so the first step is to try

$$y = mx + n \qquad (10)$$

as suggested in Procedure 2.2. If $y = mx + n$ then $\frac{dy}{dx} = m$ and $\frac{d^2y}{dx^2} = 0$ giving

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2m.$$

Thus, for (10) to be a solution of Equation (9), we require $2m = 2x + 2$ (for all x).

Unfortunately this is impossible since $2m$ is a constant where as $2x + 2$ varies with x. This shows that the differential equation does not have any particular solutions of the form $y = mx + n$. (As with the two previous examples, Procedure 2.2 breaks down here because part of the function $y = mx + n$ is a solution of the associated homogeneous equation. The part of the function which causes the trouble is $y = n$, for this is a solution of the associated homogeneous equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ and therefore does not survive substitution into the left-hand side of Equation (9).) To overcome this difficulty, I will again multiply the solution suggested in Procedure 2.2 by x and try

$$y = mx^2 + nx.$$

Then $\frac{dy}{dx} = 2mx + n$ and $\frac{d^2y}{dx^2} = 2m$, so that now

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2m + 2(2mx + n).$$

For $y = mx^2 + nx$ to be a solution we require

$$2m + 2(2mx + n) = 2x + 2,$$

that is

$$4mx + (2m + 2n) = 2x + 2.$$

This can only be true for all values of x if m and n satisfy the simultaneous equations

$$4m = 2$$
$$2m + 2n = 2.$$

Hence $m = \frac{1}{2}$ and $n = \frac{1}{2}$. Thus the differential equation has the particular solution

$$y = \frac{1}{2}(x^2 + x).$$

In each of the examples in this subsection the method described in Procedure 2.2 did not work because the solution we tried contained a term that was a solution of the associated homogeneous equation. In each case we were able to find a solution using the procedure on the next page.

> **Procedure 2.3**
>
> If the approach to finding a particular solution described in Procedure 2.2 fails to produce one, then try x times the function specified there.

Exercise 6

Find a particular solution of each of the differential equations below.

(i) $\dfrac{d^2y}{dt^2} + 4y = 2\cos 2t.$

(ii) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4e^x$

[*Solution on p. 51*]

That is very nearly the whole story. The next exercise completes it. Whether or not you try the exercise yourself, do look at its solution.

Exercise 7 (*Challenge*)

Find a particular solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x.$$

[*Solution on p. 51*]

2.4 The superposition principle

There is one final result that is relevant to finding particular solutions. I shall state the theorem using a compact notation for the left-hand side of a general second-order linear differential equation. I will write

$$L(y) = p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y$$

> **Theorem 2**
>
> Suppose that
>
> $$L(y) = f(x)$$
>
> and
>
> $$L(y) = g(x)$$
>
> are any two linear inhomogeneous differential equations, with the same associated homogeneous equation $L(y) = 0$. Suppose also that $y = u(x)$ is a solution of the first equation and $y = v(x)$ is a solution of the second equation.
>
> Then, if a and b are constants, $y = au(x) + bv(x)$ is a solution of the differential equation
>
> $$L(y) = af(x) + bg(x).$$

This result is known as the **principle of superposition**. It states that if the right-hand side of our differential equation is a linear combination of functions for which we know particular solutions, then the same linear combination of the particular solutions is what we want.

Example 9

Find a particular solution of the equation

$$\frac{d^2y}{dx^2} + 9y = 4\sin 3x + 13.$$

Solution
A particular solution of the equation

$$\frac{d^2y}{dx^2} + 9y = \sin 3x$$

is $y = -\frac{1}{6}x\cos 3x$ (see Example 7). A particular solution of

$$\frac{d^2y}{dx^2} + 9y = 1$$

is $y = \frac{1}{9}$. (Check it!) Then Theorem 2 tells us that

$$y = -\frac{4}{6}x\cos 3x + \frac{13}{9}$$

is a solution of the given differential equation.

When the right-hand side of an equation is the sum of two parts we can either solve for the two parts separately, as in Example 9, or together, as illustrated by the following example.

Example 10
To find a particular solution of the equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4 = 4e^{3t} - 7\sin 2t,$$

we would try

$$y = le^{3t} + m\cos 2t + n\sin 2t$$

(or $y = le^{3t} + \operatorname{Re}(ze^{2it})$).

Exercise 8
Find particular solutions of each of the differential equations below.

(i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4e^x - 3e^{2x}$ (Use the result of Exercise 7).

(ii) $4\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 100x = 9\cos 4t + 100.$

[*Solution on p. 51*]

2.5 General solutions

In Subsection 2.1 we discussed how the general solution of an inhomogeneous equation of the form given in Equation (1) on page 15 can be found by adding a particular solution to the complementary function. Now that we have developed a method for finding particular solutions, you should be able to find general solutions for suitable functions $f(x)$.

It is sensible when finding a general solution to find the complementary function first. This will provide advance warning of the sort of 'exceptional case' discussed in Subsection 2.3.

Exercise 9 (*Consolidation of all of Section 2*)
Find the general solution of each of the differential equations below.

(i) $\frac{d^2\theta}{dt^2} + 4\theta = 2t$

(ii) $3\frac{d^2Y}{dx^2} - 2\frac{dY}{dx} - Y = e^{2x} + 3$

(iii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^x - 5e^{2x}$

(iv) $\frac{d^2y}{dt^2} + 2\omega\frac{dy}{dt} + 2\omega^2 y = 5\sin 2\omega t + \omega^2 h$
(where ω and h are constants)

[*Solution on p. 52*]

Summary of Section 2

1. The general solution of an inhomogeneous linear differential equation is found by adding any one particular solution to the complementary function (Theorem 1). The **complementary function** is the general solution of the **associated homogeneous equation**.

2. To find a particular solution of an inhomogeneous constant coefficient equation, we try a particular expression for y, and see whether it works. The form to try for y depends on the form of the function on the right-hand side of the equation. The table below gives expressions that usually work.

$f(x)$	Try for y
$kx + l$	$mx + n$
$ke^{\alpha x}$	$me^{\alpha x}$
$k\cos\omega x + l\sin\omega x$	$\begin{Bmatrix} m\cos\omega x + n\sin\omega x \\ \text{or} \\ \operatorname{Re}(ze^{i\omega x}) \end{Bmatrix}$

3. In some cases there is no particular solution of the form suggested in the table. This occurs when the function to be tried contains a term that is a solution of the associated homogeneous equation.

   If the expression given in the table for y does not work, then try x times the given expression.

4. If the right-hand side of the differential equation is a linear combination of functions given in the table above, then a particular solution may be found by trying the same linear combination of the expressions given there for y (Theorem 2).

End of section exercises

Exercise 10

What form of expression for y should you use, in order to find a particular solution of each of the differential equations below?

(i) $\dfrac{d^2y}{dx^2} - 4y = e^{3x}$, (iii) $\dfrac{d^2y}{dx^2} - 4y = e^{-2x}$,

(ii) $\dfrac{d^2y}{dx^2} - 4y = \sin 3x$, (iv) $\dfrac{d^2y}{dx^2} + 4y = \sin 2x + 3x$.

[*Solution on p. 53*]

Exercise 11

Find the *general* solution of each of the differential equations below.

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 4$.

(ii) $\dfrac{d^2y}{dx^2} - 4y = \sin 3x$.

[*Solution on p. 53*]

3 Initial and boundary conditions (Tape Section)

In this section we will look at solutions of second-order differential equations satisfying additional conditions. In *Unit 2* you saw that the imposition of an extra condition (such as $y = 2$ when $x = 0$) on the solutions of a first-order differential equation usually picks out one particular solution by setting a value on the one arbitrary constant in the general solution.

Now the general solution of *second*-order differential equation has *two* arbitrary constants. So to pick out a particular solution from the general solution we might expect to require *two* additional conditions. This is indeed the case.

In the first part of the tape I will work through two examples which illustrate how we can find solutions satisfying given initial conditions. These examples will also provide you with an opportunity to review the techniques in Sections 1 and 2. In the second part of the tape I will discuss the uniqueness of solutions satisfying given conditions.

Start the tape when you are ready.

1

Example 1

Find the particular solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

which satisfies $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$

Solution

auxiliary equation

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = 2$$

general solution

$$\boxed{y = Ae^{x} + Be^{2x}}$$

differentiate

$$\frac{dy}{dx} = Ae^{x} + 2Be^{2x}$$

$$\boxed{\begin{array}{c} y = 0 \quad \text{when} \quad x = 0 \\ \text{so} \\ 0 = Ae^{0} + Be^{0} \end{array}}$$

$$\boxed{\begin{array}{c} \dfrac{dy}{dx} = 1 \quad \text{when} \quad x = 0 \\ \text{so} \\ 1 = Ae^{0} + 2Be^{0} \end{array}}$$

$$0 = A + B \quad (1)$$

$$1 = A + 2B \quad (2)$$

$$A = -1, \quad B = 1$$

Particular solution

$$\boxed{y = -e^{x} + e^{2x}}$$

2

Example 2

Find the particular solution of the <u>inhomogeneous</u> equation

$$\frac{d^2y}{dt^2} + 9y = e^{t} + t$$

for which $y = 0$ and $\dfrac{dy}{dt} = 1$ when $t = 0$

3 Inhomogeneous equations

1. Find the complementary function.
2. Find any particular solution.
3. Add these to get the general solution.
4. Then apply the conditions to obtain the required particular solution.

Note: the particular solutions obtained in steps 2 and 4 are not necessarily the same.

4 The general solution of $\dfrac{d^2y}{dt^2} + 9y = e^t + t$

Auxiliary equation is $\lambda^2 + 9 = 0$ so $\lambda = \pm 3i$

$$\frac{d^2y}{dt^2} + 9y = 0$$

associated homogeneous equation

$y =$

For a particular solution try

$y =$

then

$\dfrac{dy}{dt} =$

and

$\dfrac{d^2y}{dt^2} =$

so

$\dfrac{d^2y}{dt^2} + 9y = \quad = e^t + t$

so we require $l = 1/10$, $m = 1/9$ and $n = 0$ giving

$y = \frac{1}{10}e^t + \frac{1}{9}t$

particular solution

general solution

$$\boxed{y = A\cos 3t + B\sin 3t + \frac{1}{10}e^t + \frac{1}{9}t}$$

5 Applying the conditions

general solution $\longrightarrow y = A\cos 3t + B\sin 3t + \frac{1}{10}e^{t} + \frac{1}{9}t$

$y = 0$ when $t = 0$
$0 = A\cos 0 + B\sin 0 + \frac{1}{10}e^{0} + \frac{1}{9}\times 0$ $= A + \frac{1}{10}$
So $A = -\frac{1}{10}$

differentiate $\longrightarrow \frac{dy}{dt} = -3A\sin 3t + 3B\cos 3t + \frac{1}{10}e^{t} + \frac{1}{9}$

$\frac{dy}{dt} = 1$ when $t = 0$
So $B =$

So the particular solution satisfying the conditions is

$$y = \underbrace{\frac{71}{270}\sin 3t - \frac{1}{10}\cos 3t}_{\text{from complementary function}} + \underbrace{\frac{1}{10}e^{t} + \frac{1}{9}t}_{\text{original particular solution}}$$

6 Checking the solution

PROBLEM: solve $\dfrac{d^2y}{dt^2} + 9y = e^t + t$; $y = 0$, $\dfrac{dy}{dt} = 1$ when $t = 0$

SOLUTION: $\boxed{y = \frac{71}{270}\sin 3t - \frac{1}{10}\cos 3t + \frac{1}{10}e^t + \frac{1}{9}t}$

Checking the expression satisfies the equation

1. $\sin 3t$ and $\cos 3t$ are solutions of

$$\frac{d^2y}{dt^2} + 9y = 0$$

2 $y = \frac{1}{10}e^t + \frac{1}{9}t$ is a particular solution

$$\frac{d^2y}{dt^2} + 9y = \frac{1}{10}e^t + 9\left(\frac{1}{10}e^t + \frac{1}{9}t\right) = e^t + t$$

So the expression satisfies the differential equation

Check the solution satisfies the conditions

$$y = \frac{71}{270}\sin 3t - \frac{1}{10}\cos 3t + \frac{1}{10}e^t + \frac{1}{9}t$$

when $t = 0$, $y = 0 - \frac{1}{10} + \frac{1}{10} + 0 = 0$

Differentiate

$$\frac{dy}{dt} = \frac{71}{270} \times 3\cos 3t + \frac{3}{10}\sin 3t + \frac{1}{10}e^t + \frac{1}{9}$$

When $t = 0$, $\dfrac{dy}{dt} = \frac{71}{90} + 0 + \frac{1}{10} + \frac{1}{9} = 1$

Now work through Exercise 1 and check your solutions to make sure that you have understood the ideas so far.

Exercise 1

In each case find the solution of the differential equation satisfying the given conditions.

(i) $u''(t) + 4u(t) = 0$; $u(\pi/2) = 0$ and $u'(\pi/2) = 1$,

(ii) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4$; $y = 4$ and $\dfrac{dy}{dx} = -1$ when $x = 0$.

[*Solution on p. 53*]

Now try Exercise 2, which is an introduction to the next part of the tape. Its solution is covered in the tape commentary.

Exercise 2

Can you find a solution of

$$u''(x) + 4u(x) = 0$$

such that $u(0) = 0$, $u(\pi/2) = 1$?

After you have tried Exercise 2, restart the tape.

7

Answer to Exercise 2

The general solution is $u(x) = A \sin 2x + B \cos 2x$

$u(0) = 0$

$0 = A \sin 0 + B \cos 0$
so
$B = 0$

$u(\pi/2) = 1$

$1 = A \sin \pi + B \cos \pi$
so
$B = -1$

NO SOLUTION!

8

Initial conditions $\rightarrow$ $u(a) = k$ and $u'(a) = l$ $\leftarrow$ Both at <u>same</u> value of x

Boundary conditions $\rightarrow$ $u(a) = k$ and $u(b) = l$ or $u(a) = k$ and $u'(b) = l$ $\leftarrow$ at <u>different</u> values of x

9

Theorem 1: Existence and uniqueness theorem for initial condition problems

Let

$$p(x)\, u''(x) + q(x)\, u'(x) + r(x)\, u(x) = f(x)$$

be any linear second-order differential equation, for which $p(x)$ does not take the value zero for any value of x. Then there is <u>one and only one</u> solution of this differential equation satisfying the pair of initial conditions:

$$\boxed{u(a) = k \quad \text{and} \quad u'(a) = l}$$

Exercise 3

Identify each of the following as either an initial condition problem, or a boundary condition problem. Solve each problem.

(i) $u''(x) + 4u(x) = 0$; $u(0) = 1$, $u'(0) = 0$.

(ii) $u''(x) + 4u(x) = 0$; $u(0) = 0$, $u(\pi/2) = 0$.

(iii) $u''(x) + 4u(x) = 0$; $u(0) = 0$, $u'(0) = 0$.

(iv) $u''(x) + 4u(x) = 0$; $u(-\pi) = 1$, $u(\pi/4) = 2$.

[*Solution on p. 53*]

These examples bring up some points of interest. Boundary condition problems may be 'well-behaved', in that they produce a unique particular solution, as in (iv) above. Indeed, this is often the case. However, a boundary condition problem may have no solution, as we saw in the tape, or it may have infinitely many solutions, as in (ii) above.

For a *homogeneous* equation, with initial conditions of the form $u(a) = 0$, $u'(a) = 0$ there is no need to do any detailed calculation, as the solution must be the zero function (c.f. example (iii) above. This short cut cannot be applied to boundary condition problems, as there is no guarantee that the zero function is the only solution (c.f. example (ii) above). Nor can it be applied to *inhomogeneous* equations, for then the zero function is not a solution at all.

Exercise 4

Solve each of the following problems.

(i) $\dfrac{d^2y}{dx^2} - y = 0$; $y = 0$ and $\dfrac{dy}{dx} = 0$ at $x = 0$.

(ii) $\dfrac{d^2y}{dx^2} - y = 8$; $y = 0$ and $\dfrac{dy}{dx} = 0$ at $x = 0$.

(iii) $\dfrac{d^2x}{dt^2} + 2\omega\dfrac{dx}{dt} + \omega^2 x = 0$; $x = a$ and $\dfrac{dx}{dt} = 0$ at $t = 0$,

where ω and a are given constants. (You can use Exercise 2(i) of Section 1 here, if you wish.)

[*Solution on p. 54*]

Exercise 5 (*Challenge—optional*)

(i) There is an analogue of Theorem 1 for first-order equations. What would you expect it to say?

(ii) Can you find any solution(s) of

$$x\frac{dy}{dx} - y = 0$$

satisfying $y = 0$ when $x = 0$?

(iii) How does the answer to (ii) relate to your 'existence and uniqueness' theorem in (i)?

[*Solution on p. 54*]

Summary of Section 3

In this section we examined the problem of finding solutions which, in addition to satisfying a second-order differential equation, also satisfy two extra conditions.

Such conditions are called **initial conditions** if they both occur at the *same* value of the independent variable. Otherwise they are called **boundary conditions**.

For initial conditions we have an existence and uniqueness theorem which is stated in Frame 9.

For boundary conditions there is no guarantee that a solution will exist. Even if a solution does exist, it is not necessarily unique.

End of section exercise

Exercise 6

Find the solutions (if any) of each of the following problems.

(i) $u''(t) + 4u'(t) + 5u(t) = 0; \quad u(0) = 0, \quad u'(0) = 2.$

(ii) $u''(t) + 9u(t) = 0; \quad u(0) = 0, \quad u'(\pi/3) = 1.$

(iii) $\dfrac{d^2y}{dx^2} + \dfrac{2dy}{dx} + 2y = 0; \quad y = 0, \quad \dfrac{dy}{dx} = 0 \text{ at } x = \pi.$

State in each case whether it is an initial or a boundary condition problem.

[*Solution on p. 54*]

4 Oscillations and graphs (Television Section)

4.0 Introduction

In this section we study the graphs of certain functions that arise as solutions of second-order differential equations. We will concentrate on cases of importance in dynamics, particularly those where the solutions contain sinusoidal terms, and so are oscillatory. Many of the ideas discussed in the programme are also covered in the first three sections of the unit. Those ideas introduced in the programme, and not covered elsewhere in the unit, are summarized in Subsection 4.2.

4.1 Preliminary work

The programme divides into three parts.

1. We study the differential equation

$$\frac{d^2y}{dx^2} - ky = 0. \tag{1}$$

If k is positive the solution of Equation (1) involves exponential functions, but if k is negative it is sinusoidal. The programme reviews some ideas associated with sinusoids that were introduced in *Unit 5*. It also examines the relation of the amplitude and phase of the solution to initial conditions specified on the solution and its derivative.

2. We study the homogeneous equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0. \tag{2}$$

(The algebraic forms of the coefficients are chosen because they provide simplicity later.) We know that the solution of Equation (2) is oscillatory (in fact purely sinusoidal) if $\alpha = 0$, because we then have Equation (1) with k negative. For $0 < \alpha < \omega$ the solution is an oscillation whose amplitude decreases with increasing x, but if $\alpha > \omega$, the solution is not oscillatory.

Either way, so long as $\alpha > 0$, the solutions decrease to zero as x increases.

3. We study an inhomogeneous equation:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin 2x. \tag{3}$$

Any solution of Equation (3) is the sum of two parts. One part is sinusoidal with the same period as $\sin 2x$, and is the same for all solutions of the equation. The other part is a solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

This part is an oscillation whose amplitude decreases as x increases, and becomes very small as x becomes large. (This second part of the solution may therefore be

called *transient*.) For large x only the first part of the solution remains (and so this part is sometimes called the ***steady-state solution***).

To gain maximum benefit from the programme you will need to have studied the preceding sections of the unit. If you have not, you will find many of the ideas from these sections reviewed in the programme, but you will probably find that the pace of the programme is too fast for you to follow the details. Either way, remember that the important things to concentrate on are the shapes of the graphs of the solutions of Equations (1) and (2), and the way a solution of Equation (3) can be divided into two parts.

The programme makes some use of the ideas concerning sinusoidal oscillations and phasors that where introduced in *Unit 5*. Exercise 1 serves to remind you of these. Before viewing the programme, work through this, and, if you have studied Section 1 of the unit, Exercise 2 as well.

Exercise 1

Find the phasor of the sinusoidal function

$$f(t) = 4\cos 5t - 3\sin 5t,$$

and hence find the amplitude and phase of this oscillation.

[*Solution on p. 55*]

Exercise 2

Solve the auxiliary equation of the homogeneous differential equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0.$$

For what range of values of α and ω are the roots real?

[*Solution on p. 55*]

As soon as possible after viewing the programme, read the notes in Subsection 4.2 and then work through Exercises 3, 4 and 5. The remaining exercises may be left till later if you wish. If you have not studied the earlier parts of the unit it is still advisable to study the notes and try Exercises 3, 4 and 5 after viewing the programme, although there will be points that you will need to return to after you have studied the first three sections of the unit.

Now watch the television programme 'Good vibrations'.

TV6

4.2 Good vibrations

This subsection summarizes the new material in the television programme that you will be expected to know, and also takes some of the ideas a little further.

1. An equation that will be useful in *Unit 7* is

$$\frac{d^2y}{dx^2} + \omega^2 y = 0. \tag{4}$$

The general solution of this equation (see Section 1) is

$$y = C\cos\omega x + D\sin\omega x,$$

where C and D are arbitrary constants. You saw in *Unit 5*, Subsection 4.3, that this can also be written as

$$y = A\cos(\omega x + \phi). \tag{5}$$

Thus (5), with A and ϕ arbitrary constants, is an alternative way of writing the general solution of Equation (4). This solution is a sinusoidal oscillation (see Figure 1). The constant ω in (5) is called the **angular frequency** of the sinusoid, and is determined by the differential equation itself. The amplitude A and phase ϕ of the oscillation depend on the initial conditions (the values of y and $\dfrac{dy}{dx}$ at $x = 0$).

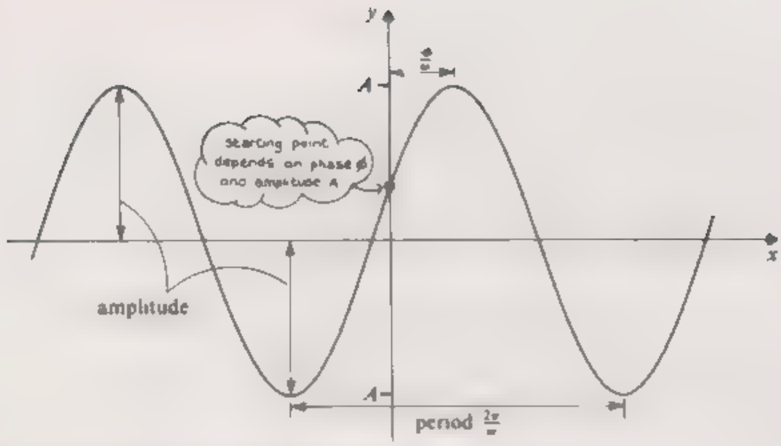


Figure 1. The graph of the sinusoidal oscillation $y = A\cos(\omega t + \phi)$

Fitting different initial conditions will have the effect of expanding or contracting vertically the graph in Figure 1 (varying A), and shifting the curve sideways (varying ϕ) as in Figure 2.

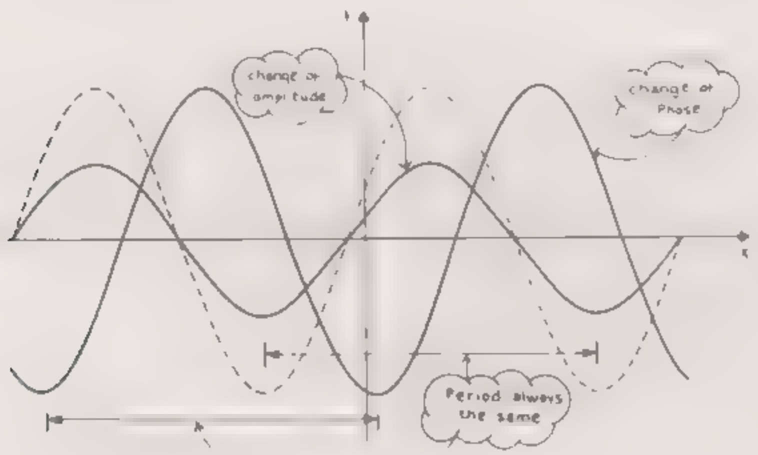


Figure 2. The effect of varying the constants A or ϕ in Figure 1

2. An equation that will be useful in *Unit 8* is

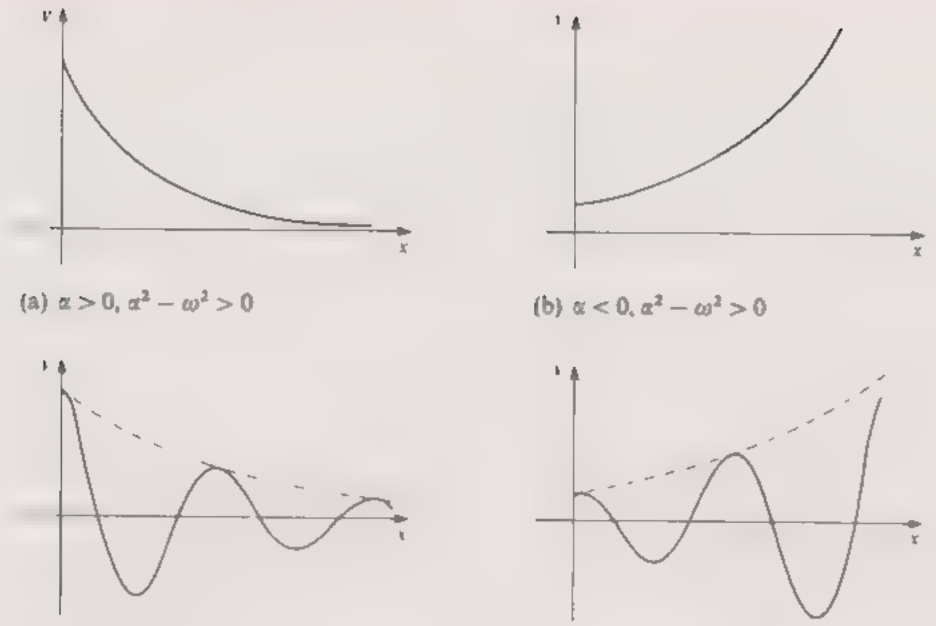
$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0. \tag{6}$$

The nature of the solutions of this equation depends on the signs of α and $\alpha^2 - \omega^2$. Of particular importance for dynamical applications is the case when α is positive. The fundamental nature of the solution in four possible cases is summarized below.

$\alpha^2 - \omega^2$ \ α	positive	negative
positive	Decreasing exponential	Increasing exponential
negative	Oscillation with decreasing amplitude	Oscillation with increasing amplitude

See the solution to Exercise 12 at the end of this section for an explanation of the entries in this table.

The graphs of typical solutions in each of these four cases are shown in Figure 3.



(a) $\alpha > 0$, $\alpha^2 - \omega^2 > 0$

(b) $\alpha < 0$, $\alpha^2 - \omega^2 > 0$

(c) $\alpha > 0$, $\alpha^2 - \omega^2 < 0$

(d) $\alpha < 0$, $\alpha^2 - \omega^2 < 0$

Figure 3. Graphs of various solutions of Equation (6)

Let us concentrate for a moment on the case when $\alpha > 0$, $\alpha^2 - \omega^2 > 0$ (Figure 3(a)). In this case the general solution of Equation (6) is a linear combination of two decreasing exponentials. By choosing different linear combinations we can find solutions fitting different initial conditions. These solutions have graphs which are variants of the graph in Figure 3(a). Figure 4 shows the sort of graphs that we can obtain in this case. All the solutions have the property of tending to zero as x becomes large.

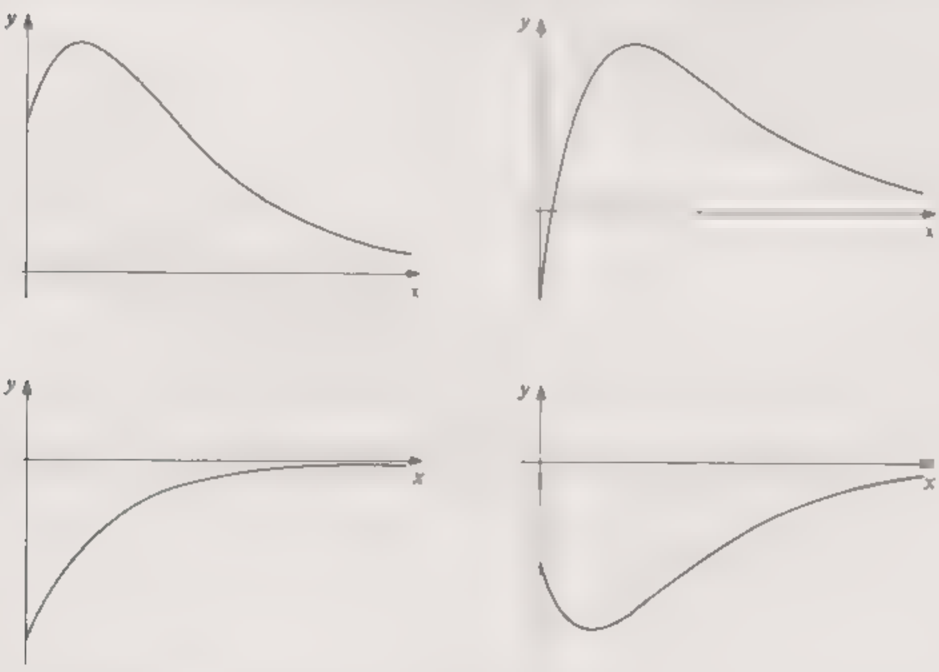


Figure 4. Graphs of possible solutions of Equation (6) when $\alpha > 0$ and $\alpha^2 - \omega^2 > 0$, fitting various initial conditions.

In the cases where $\alpha < 0$, $\alpha^2 - \omega^2 > 0$, we get solutions showing similar variations on the basic increasing exponential shape shown in Figure 3(b). In the oscillatory cases (Figure 3(c) and 3(d)), the general solution of Equation (6) can be written in the form $y = Ae^{-\alpha x}\cos(\nu x + \phi)$ where $\nu = \sqrt{\omega^2 - \alpha^2}$ and A and ϕ are arbitrary constants. Changing the values of A and ϕ will alter the initial conditions satisfied by the solution, but the solution curve will have the same basic shape.

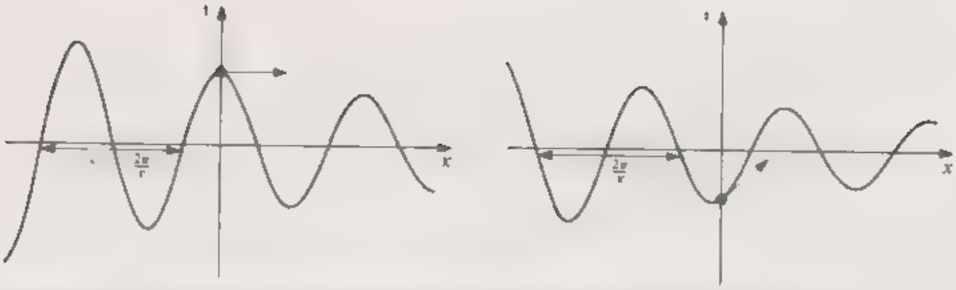
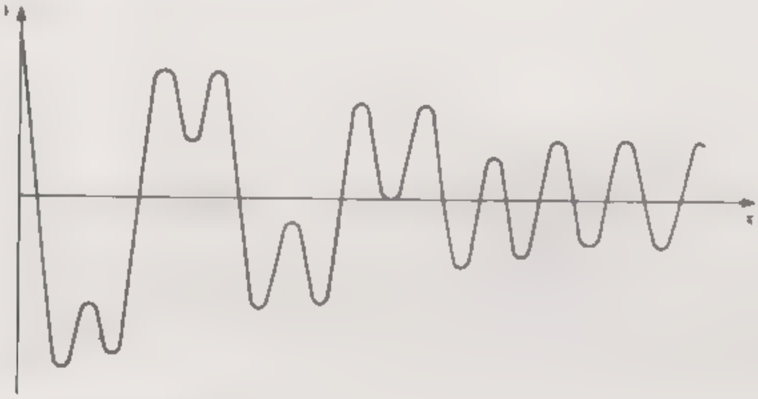


Figure 5 In oscillatory cases, changing the initial conditions changes the values of the arbitrary constants A and ϕ. The period remains the same.

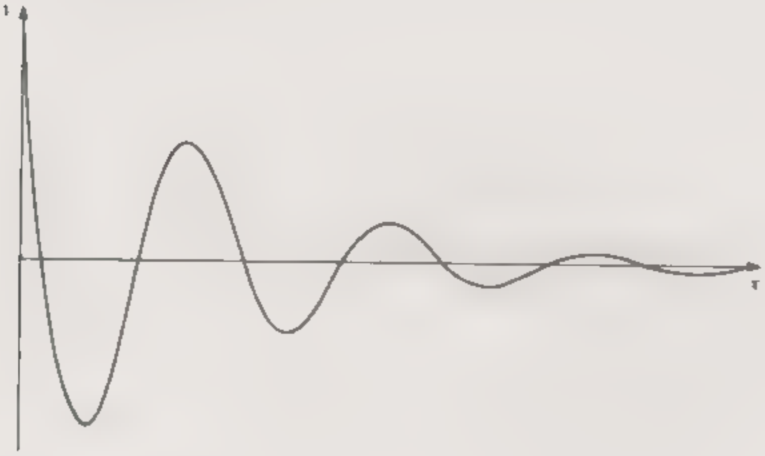
3. We saw in the programme how a solution of

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5y = \sin 2x \tag{7}$$

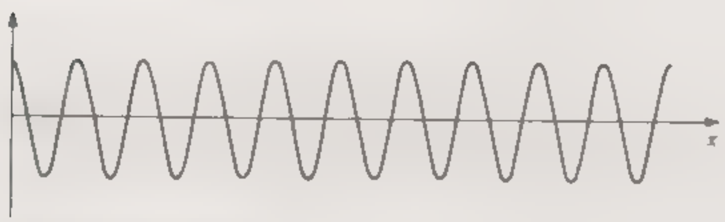
is the sum of two parts (see Figure 6).



(a) A solution of Equation (7)



(b) Its transient part



(c) Its steady state part

Figure 6. The solution (a) is the sum of two parts, shown in (b) and (c)

One part is a decreasing oscillation (Figure 6(b)), the other is a pure sinusoid (Figure 6(c)). For sufficiently large x the solution becomes effectively equal to the sinusoidal part. Whatever initial conditions we apply to Equation (7), this sinusoidal part of the solution is *exactly* the same. (That is, its amplitude, phase and frequency do not depend on the initial conditions.) The initial conditions affect only the decreasing part of the solution.

The decreasing part of the solution is called a **transient**. The sinusoid to which the solution settles when x is large is called the **steady-state solution** of Equation (7).

These ideas apply more generally. The *general solution* of

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = k\cos \nu x + l\sin \nu x \tag{8}$$

is the sum of the *complementary function*—the solution of

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0, \tag{9}$$

and any one *particular solution* of Equation (8) (see Section 2). We have seen that if $\alpha > 0$ the solutions of Equation (9) tend to zero as x becomes large. In other words, if $\alpha > 0$ the contribution from the complementary function is a transient. There is one sinusoidal particular solution of Equation (8), and this is the **steady-state solution**. Any other solution of Equation (8) is the sum of a **transient** (from the complementary function) and this steady-state solution. The steady-state solution does *not* depend on the initial conditions. Only the transient term is affected by the initial conditions.

4. Since the steady-state solution of Equation (8) (with $\alpha > 0$) is a sinusoidal function, it can be represented by a phasor. The easiest way to find the steady-state solution is to calculate this phasor directly. The procedure for doing this is discussed in Subsection 2.3, and also in Example 1 below. Example 1 also illustrates how the phasor can be used to find the amplitude and phase of the steady-state solution. The angular frequency of the steady-state solution of Equation (8) is ν, the angular frequency of the sinusoid on the right-hand side of the equation.

Example 1

Find the amplitude, phase and angular frequency of the steady-state solution of

$$4\frac{d^2y}{dx^2} + 9\frac{dy}{dx} + 100y = 9\cos\frac{25}{4}x.$$

Use the results to write down the steady-state solution.

Solution

The steady-state solution is a sinusoid with the same angular frequency as the sinusoid on the right-hand side of the equation, that is

$$\text{angular frequency} = \frac{25}{4}.$$

We are therefore looking for a solution of the form $C\cos\frac{25}{4}x + D\sin\frac{25}{4}x$ or equivalently $A\cos(\frac{25}{4}x + \phi)$. Since we are interested in the amplitude and phase we want the solution in the second form here. To do this we shall find the phasor z of this sinusoid by trying $y = \mathrm{Re}(ze^{25ix/4})$, which leads (by Procedure 2.2(a)) to

$$\left(4\left(\frac{25}{4}i\right)^2 + 9\left(\frac{25}{4}i\right) + 100\right)z = 9.$$

Solving for z gives

$$z = -\frac{2}{25}(1 + i).$$

The amplitude and phase of the oscillation are given by the amplitude and phase of this phasor, that is

$$\text{amplitude} = |z| = \frac{2\sqrt{2}}{25}$$

$$\text{phase} = \text{Arg } z = \frac{-3\pi}{4}$$

The steady-state solution is therefore

$$y = \frac{2\sqrt{2}}{25}\cos\left(\frac{25}{4}x - \frac{3\pi}{4}\right).$$

Exercises 3, 4 and 5 are to be worked directly after watching the programme.

Exercise 3

Describe (in one sentence) the graph of any solution of the differential equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0.$$

[*Solution on p. 55*]

Exercise 4

(i) What conditions must α and ω satisfy if the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0$$

is to be a decaying oscillation?

(ii) Sketch the graphs of the solutions of each of the differential equations below, that satisfy the specified conditions.

(a) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 9y = 0;\qquad y = 4$ and $\dfrac{dy}{dx} = 0$ at $x = 0$.

(b) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 4y = 0;\qquad y = 2$ and $\dfrac{dy}{dx} = 1$ at $x = 0$.

(Do not solve the equations. Your sketches should show (*I*) behaviour near $t = 0$, (*II*) behaviour for large t, (*III*) whether or not the solution is oscillatory.)

[*Solution on p. 55*]

Exercise 5

Describe the solutions of the equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = \cos vx,$$

where $0 < \alpha < \omega$. In particular, comment on the long-term behaviour of the solution, and the nature of the part of the solution that is dependent on the initial conditions.

[*Solution on p. 55*]

The following exercises are to be worked after viewing the programme, and after working the first four sections of the unit.

Exercise 6

Find the amplitude, phase and angular frequency of the particular solution of

$$\frac{d^2y}{dx^2} + 9y = 0$$

satisfying $y = 4$ and $\dfrac{dy}{dx} = 9$ at $x = 0$.

[*Solution on p. 55*]

Exercise 7

Consider the differential equation

$$u''(x) + \omega^2 u(x) = 0$$

with the boundary conditions $u(a) = k$ and $u\left(a + \frac{2\pi}{\omega}\right) = l$. Do these two conditions guarantee a unique solution? (Consider the cases $k = l$ and $k \neq l$ separately.)

Compare this with Theorem 1 of Section 3.

[*Solution on p. 55*]

Exercise 8

Find the particular solution of

$$\frac{d^2x}{dt^2} + 4\omega\frac{dx}{dt} + \omega^2 x = 0$$

satisfying the conditions $x = a$ and $\frac{dx}{dt} = 0$ at $t = 0$.

(Use the general solution of this equation found in Example 7 of Section 1.)

[*Solution on p. 55*]

Exercise 9

Find the particular solution of

$$\frac{d^2x}{dt^2} + 2\beta\omega\frac{dx}{dt} + \omega^2 x = 0,$$

where $0 < \beta < 1$ and $0 < \omega$, that satisfies $x = a$ and $\frac{dx}{dt} = 0$ at $t = 0$.

Sketch the graph of this solution.

[*Solution on p. 56*]

Exercise 10

Find the amplitude, phase and angular frequency of the steady-state solution of

$$4\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 100x = 9\cos 5t.$$

[*Solution on p. 56*]

Exercise 11

Find the phasor of the steady-state solution of

$$\frac{d^2x}{dt^2} + 2\beta\omega\frac{dx}{dt} + \omega^2 x = \cos \nu t$$

where β and ω are positive constants, and $\beta < 1$. Find the amplitude of this solution.

[*Solution on p. 56*]

Exercise 12 (*Optional*)

Explain why the graphs of the solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0$$

take the forms specified in the table on p. 36.

[*Solution on p. 56*]

5 A numerical method

5.0 Introduction

All the second-order differential equations you have encountered in this unit so far have been amenable to analytical solution, in that we have been able to express the solution in terms of a formula. Unfortunately, not all second-order differential equations which occur in practice can be solved analytically. For example the differential equation

$$\frac{d^2y}{dt^2} = -10\sin y,$$

which arises in the study of the large oscillations of a pendulum, has no solution in terms of the elementary functions such as trigonometric or exponential functions.

In such a situation we may resort to using a numerical method. In *Unit 2* we used Euler's method to solve first-order differential equations of the form

$$\frac{dy}{dx} = m(x, y).$$

This involves using a recurrence relation to generate values y_r that give approximations to $y(x_r)$, where the difference $x_{r+1} - x_r$ between successive values of x is a constant h (the step length). In Euler's method we approximate the derivative of y at $x = x_r$ by $\frac{y_{r+1} - y_r}{h}$, to obtain the recurrence relation

$$y_{r+1} = y_r + hm(x_r, y_r).$$

In this section we will show how an extension of this method can be used to solve second-order differential equations.

5.1 Euler's method

There are several methods of obtaining numerical approximations to the solution of a second-order differential equation. The method I will describe here is to replace the second-order equation by two first-order differential equations, and to apply Euler's method to each of these. (Later in the course, we will see other, and better, methods than Euler's for first-order equations. Use of these, together with the basic idea of replacing a second-order equation by two first-order ones, will provide more satisfactory methods for second-order equations.)

The key idea is to introduce another variable z to represent the first derivative of y. That is, we write

$$z = \frac{dy}{dx}.$$

If we now differentiate z, we obtain

$$\frac{dz}{dx} = \frac{d^2y}{dx^2}.$$

To see how this idea is used we look at the following example.

Example 1

Replace the second-order differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

by a pair of first-order equations of the form

$$\frac{dy}{dx} = z$$

and

$$\frac{dz}{dx} = \text{(some function of } x, y \text{ and } z \text{ not involving their derivatives)}$$

Solution
We can rewrite the differential equation as

$$\frac{dz}{dx} - 5\frac{dy}{dx} + 6y = 0.$$

Also, replacing $\frac{dy}{dx}$ by z, we obtain

$$\frac{dz}{dx} - 5z + 6y = 0.$$

Then the pair of first-order equations (written in a form suitable for applying Euler's method) is

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = 5z - 6y.$$

Similarly any linear differential equation of the form

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = f(x)$$

(for which $p(x)$ is non-zero for all x) can be written in the form

$$\left.\begin{aligned}\frac{dy}{dx} &= z\\ \frac{dz}{dx} &= m(x, y, z)\end{aligned}\right\} \qquad (1)$$

where $m(x, y, z)$ is some function of the three variables x, y and z not involving their derivatives. Many non-linear equations can also be treated in this way.

Exercise 1
Replace each of the following second-order differential equations by a pair of first-order equations of the form given in Equation (1).

(i) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \cos x$

(ii) $\frac{d^2y}{dt^2} + 32\sin y = 0$

[*Solution on p. 57*]

Having changed our second-order differential equation into a pair of first-order equations, we can set up the equations for Euler's Method.

Example 2
Formulate Euler's method for solving the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0. \qquad (2)$$

Solution
From Example 1 we can replace the equation by the two first-order equations

$$\frac{dy}{dx} = z \qquad (3)$$

$$\frac{dz}{dx} = 5z - 6y. \qquad (4)$$

To apply Euler's method to Equation (3), we approximate the derivative of y at $x = x_r$ by

$$\frac{y_{r+1} - y_r}{h}. \tag{5}$$

Here y_r is Euler's approximation to $y(x_r)$, the value of the true solution at x_r. The values of x_r are given by

$$x_r = x_0 + rh.$$

Replacing $\frac{dy}{dx}$ in Equation (3) by the Expression (5) gives the recurrence relation

$$y_{r+1} = y_r + hz_r,$$

where z_r is the approximation to $z(x_r)$.

Treating Equation (4) in a similar way we obtain

$$z_{r+1} = z_r + h(5z_r - 6y_r).$$

Thus Euler's method applied to the Second-order equation (2) yields the pair of recurrence relations:

$$y_{r+1} = y_r + hz_r$$
$$z_{r+1} = -6hy_r + (1 + 5h)z_r.$$

Exercise 2

Derive the recurrence relations for Euler's method applied to the differential equations

(i) $\frac{dy}{dx} = z$

$\frac{dz}{dx} = -4z - 5y + \cos x$

(ii) $\frac{dy}{dx} = z$

$\frac{dz}{dx} = -32 \sin y$

[*Solutions on p. 57*]

We are now almost ready to carry out some computations. All that remains is to see how the initial conditions fit into this scheme. Although my main purpose in developing a numerical method is to be able to attack equations that *cannot* be solved analytically, I shall deliberately illustrate the method with an equation that we *can* solve analytically. We will then be able to see how accurate the method is.

Example 3

Use Euler's Method to obtain the value of y when $x = 1$ using a step length $h = 0.1$ for the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

with the initial conditions $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 0$.

Method

We saw in Example 2 that Euler's method gives rise to the recurrence relations

$$y_{r+1} = y_r + hz_r$$
$$z_{r+1} = -6hy_r + (1 + 5h)z_r.$$

The initial conditions can be written as $y_0 = 1$ and $z_0 = 1$. Thus with $h = 0.1$ we have

$$\left.\begin{aligned} y_{r+1} &= y_r + 0.1z_r \\ z_{r+1} &= -0.6y_r + 1.5z_r \end{aligned}\right\} \quad (y_0 = 1 \text{ and } z_0 = 1).$$

We can now use the recurrence relations to calculate y_1 and z_1, then y_2 and z_2, and so on.

The following table (in which the results have been calculated to 3 decimal places) shows the results of the calculation.

x_r	y_r	z_r	$y_r + 0.1z_r$	$0.6y_r + 1.5z_r$
0.0	1.000	1.000	1.100	0.900
0.1	1.100	0.900	1.190	0.690
0.2	1.190	0.690	1.259	0.321
0.3	1.259	0.321	1.291	−0.274
0.4	1.291	−0.274	1.264	−1.186
0.5	1.264	−1.186	1.145	−2.537
0.6	1.145	2.537	0.891	4.493
0.7	0.891	−4.493	0.442	7.274
0.8	0.442	7.274	0.285	−11.176
0.9	0.285	−11.176	1.403	
1.0	−1.403			

5.2 Accuracy of the method

For the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0; \qquad y = 1 \text{ and } \frac{dy}{dx} = 1 \text{ when } x = 0$$

considered in the previous subsection we can compare our numerical results with the analytical solution, which is

$$y = 2e^{2x} - e^{3x}.$$

The following graph gives both the analytical and numerical results.

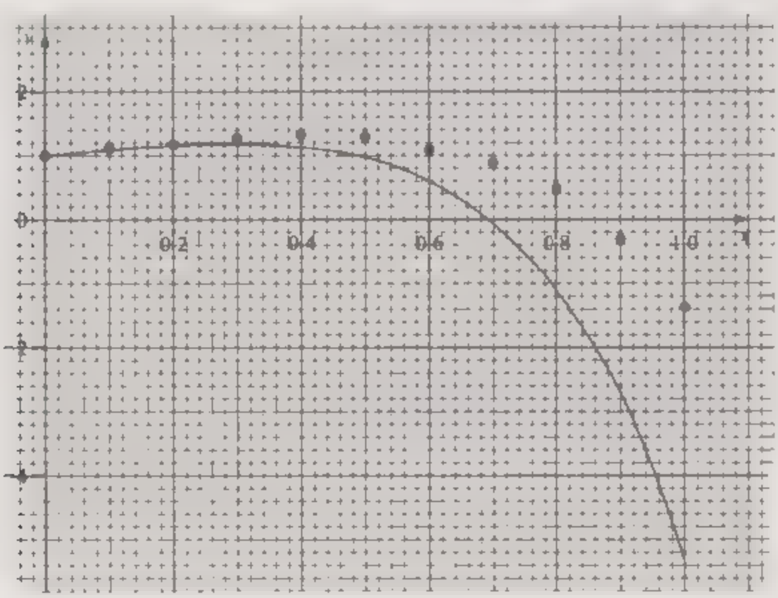


Figure 1

These results are not particularly good for this value of h though it can be seen that the numerical results do follow the general shape of the solution. With smaller step sizes we would expect the results to be much closer to the true solution (provided we work to enough decimal places). However, the smaller the step-size the more work we have to do to evaluate y at $x = 1$. Thus in general we would

wish to use a computer to obtain our results. The following table gives the approximation to $y(1)$ obtained for various step sizes h.

h	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{40}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{1000}$
$y(1)$	-1.402	-2.911	-3.585	3.964	-4.207	4.729	-5.247

The correct value for y at $x = 1$ is $2e^2 - e^3 = -5.307$.

Exercise 3

(i) Continue the table below to find an approximation to $y(3.2)$, where y is the solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

satisfying the initial conditions $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$

x_r	y_r	z_r	$y_r + 0.2z_r$	$-0.2y_r + z_r$
0	0	1.0	0.2	1.0
0.2	0.2	1.0	0.4	0.96
0.4	0.4	0.96	0.592	0.88
0.6	0.592	0.88	0.768	0.762
0.8	0.768	0.762	0.920	0.608
1.0	0.920	0.608	1.042	0.424
1.2	1.042	0.424	1.127	0.216
1.4	1.127	0.216	1.170	−0.009
1.6	1.170	−0.009	1.168	−0.243
1.8	1.168	−0.243	1.119	−0.477
2.0	1.119	−0.477	1.024	−0.701
2.2	1.024	−0.701	0.884	−0.906
2.4	0.884	−0.906	0.703	−1.083
2.6	0.703	−1.083	0.486	
2.8	0.486			
3.0				
3.2				

(ii) What is the exact solution of this equation satisfying the given initial conditions?

(iii) Compare the results of Euler's method, given in the solution to part (i) above, with the true solution.

[*Solution on p. 57*]

Although the results in Exercise 3 are not very accurate, the values obtained for y_r do follow the general pattern of the solution. It would be reassuring if this were always the case, but unfortunately this is not so. This can be seen in the next exercise.

Exercise 4

Apply Euler's method to find $y(1)$ using a step length $h = 0.2$ for the differential equation

$$\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 20y = 0$$

with $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 0$. Compare your results with the true solution.

[*Solution on p. 57*]

The reason for the behaviour shown by the application of Euler's method in Exercise 4 is discussed in the unit on numerical solutions of differential equations.

Summary of Section 5

Euler's method for obtaining numerical approximations to the solution of a second-order differential equation is to put

$$z = \frac{dy}{dx}$$

and replace the second-order differential equation by a pair of first-order equations. Euler's method as described in *Unit 2* can then be applied to each of these first-order equations.

End of section exercise

Exercise 5

Use Euler's method with step length $h = 0.2$ to find an approximation to $y(1)$, where y satisfies

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2$$

with $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$. Compare your results with a graph of the analytic solution.

[*Solution on p. 58*]

6 End of unit exercises

Exercises 1, 2 and 3 provide further practice in the main skill that this unit aims to teach — the solution of second-order constant-coefficient differential equations. You should try to make sure that you are confident on this point. If you are, you may like to try some of the other exercises in this section, which use the material in the unit, but do not concentrate solely on this central skill.

Exercise 1

For each of the equations below find the solution satisfying the given initial conditions.

(i) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$; with $y = 2$ and $\frac{dy}{dx} = 1$ at $x = 0$

(ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = e^{2x}$; with $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$.

(iii) $\frac{d^2y}{dt^2} - 9y = 3\sin 3t + 18$; with $y = 0$, and $\frac{dy}{dt} = 0$ at $t = 0$.

(iv) $\frac{d^2\theta}{du^2} = \frac{1}{u^2}$ $(u > 0)$; with $\theta = 1$ and $\frac{d\theta}{du} = 2$ at $u = 1$.

[*Solution on p. 58*]

Exercise 2

Find the general solution of

(i) $\frac{d^2y}{dx^2} + 16y = 4\cos 4t + 8\sin 4t$;

(ii) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 3e^{-2t} + 8t$.

[*Solution on p. 59*]

Exercise 3

For each of the following differential equations sketch the graph of the solution satisfying the given initial conditions.

(i) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0$; with $y = -1$ and $\frac{dy}{dx} = -2$ at $x = 0$

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$; with $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$.

[*Solution on p. 59*]

Exercise 4

Suppose that x denotes the position of an object at time t, and that x satisfies the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = 2 + 2\sin 3t.$$

(i) Sketch a graph showing the variation of position with time, a long time after the object is set in motion. Does the value of x ever become negative during this part of the motion?

(ii) Can you make more precise the meaning of the phrase 'a long time after the object is set in motion' in part (i).

[*Solution on p. 59*]

Exercise 5

For what values of the constant λ can a solution other than the zero function be found for the boundary condition problem

$$u''(x) + \lambda^2\pi^2u(x) = 0; \qquad u(0) = 0, \quad u(1) = 0?$$

[*Solution on p. 60*]

Exercise 6

This exercise looks at an application of the theoretical ideas introduced in this unit (in particular, Theorem 1 of Section 3). It concerns a possible approach to the formal definition of the trigonometric functions.

We define the functions S and C to be the solutions of the differential equation

$$u''(x) + u(x) = 0 \tag{1}$$

satisfying the conditions,

$$S(0) = 0 \quad S'(0) = 1$$

and

$$C(0) = 1 \quad C'(0) = 0.$$

respectively.

(i) Explain why this is sufficient to define the functions S and C.

(ii) Prove: (a) that S' is a solution of Equation (1); (b) that $S'(0) = 1$ and $S''(0) = 0$. Deduce that $S' = C$.

(iii) Use an argument similar to that in (ii) to prove that $C' = -S$

(iv) Find the derivative of $C^2 + S^2$, and deduce that

$$C^2 + S^2 = 1.$$

[*Solution on p. 60*]

We could continue to deduce other properties of the trigonometric functions in the same sort of way, but this is sufficient to give you the flavour of the approach.

Appendix

Solution to the exercise in the Introduction

1. (i) Equations (a), (c) and (d) are second-order. Equation (b) is first-order.

   (ii) Equations (a), (b) and (d) are linear.

   (iii) Equations (b) and (c) have constant coefficients.

   (iv) Equations (a) and (c) are homogeneous.

Solutions to the exercises in Section 1

1. The solutions are given in the following table:

	Auxiliary equation	Roots	Case (Procedure 1.1)	General solution
(i)	$\lambda^2 - 6\lambda + 5 = 0$	$\lambda = 1, 5$	(i)	$y = Ae^x + Be^{3x}$
(ii)	$\lambda^2 + 9 = 0$	$\lambda = \pm 3i$	(iii)	$\theta = A\cos 3t + B\sin 3t$
(iii)	$\lambda^2 - 4 = 0$	$\lambda = \pm 2$	(i)	$z = Ae^{2u} + Be^{-2u}$
(iv)	$\lambda^2 + 2\lambda + 1 = 0$	$\lambda = -1$ only	(ii)	$y = Ae^{-x} + Bxe^{-x}$
(v)	$\lambda^2 + 4\lambda + 8 = 0$	$\lambda = -2 \pm 2i$	(iii)	$u = e^{-2t}(A\cos 2t + B\sin 2t)$
(vi)	$2\lambda^2 + 3\lambda = 0$	$\lambda = 0, -3/2$	(i)	$y = A + Be^{-3x/2}$

In each case A and B are arbitrary constants.

2. (i) The auxiliary equation is

$$\lambda^2 + 2\omega\lambda + \omega^2 = 0$$

i.e. $(\lambda + \omega)^2 = 0$.

This is Case (ii), with the only root $\lambda = -\omega$. The general solution is therefore

$$y = e^{-\omega x}(Ax + B),$$

where A and B are arbitrary constants.

(ii) The auxiliary equation is

$$5\lambda^2 + 6\omega\lambda + 5\omega^2 = 0$$

thus

$$\lambda = \frac{-6\omega \pm \sqrt{36\omega^2 - 100\omega^2}}{10}$$

$$= \frac{\omega}{5}(-3 \pm 4i).$$

This is Case (iii) of Procedure 1.1, and the general solution is therefore

$$x = \exp\left(-\frac{3}{5}\omega t\right)\left(A\cos\frac{4}{5}\omega t + B\sin\frac{4}{5}\omega t\right),$$

where A and B are arbitrary constants.

3. If we can find two linearly independent solutions, then we can use Theorem 2 to write down the general solution since, with the domain $(x > 0)$, the differential equation is of the form specified in the Theorem. To find some solutions, we try $y = x^n$, as suggested. If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$ and $\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$. Thus

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2x^2 n(n-1)x^{n-2} - xnx^{n-1} + x^n$$

$$= x^n[2n(n-1) - n + 1]$$

$$= (2n^2 - 3n + 1)x^n.$$

This will equal zero (for all x) if and only if

$$2n^2 - 3n + 1 = 0$$

That is, if

$$(2n - 1)(n - 1) = 0$$

Hence $n = \frac{1}{2}$ or 1. So $y = x$ and $y = x^{1/2}$ are solutions of the given differential equation, and these are linearly independent. By Theorem 2 the general solution is

$$y = Ax + Bx^{1/2} \qquad (x > 0)$$

4.(i) The auxiliary equation is $3\lambda - 1 - 2\lambda^2 = 0$.

(ii) This can be rewritten as

$$2\lambda^2 - 3\lambda + 1 = 0$$

i.e. $(2\lambda - 1)(\lambda - 1) = 0$

so $\lambda = 1$ or $\frac{1}{2}$.

(iii) By Procedure 1.1 the general solution is

$$y = Ae^x + Be^{\frac{1}{2}x}$$

where A and B are arbitrary constants.

5. The solutions are given in the following table:

	Auxiliary equation	Roots	Case (Procedure 1.1)	General solution
(i)	$\lambda^2 + 2\lambda + 2 = 0$	$-1 \pm i$	(iii)	$y = e^{-x}(A\cos x + B\sin x)$
(ii)	$\lambda^2 - 16 = 0$	± 4	(i)	$y = Ae^{4x} + Be^{-4x}$
(iii)	$\lambda^2 - 4\lambda + 4 = 0$	2 (only)	(ii)	$y = Ae^{2x} + Bxe^{2x}$
(iv)	$\lambda^2 = 0$	0 (only)	(ii)	$\theta = A + Bt$

In each case A and B are arbitrary constants.

6. The general solution of the differential equation contains sine and cosine terms only in case (iii) of Procedure 1.1, when the solutions of the auxiliary equation are complex. The auxiliary equation is

$$\lambda^2 + 4k\lambda + 4 = 0$$

which has complex solutions when

$$16k^2 - 16 < 0$$

i.e. $k^2 < 1$

i.e. $-1 < k < 1$.

7. No. For Theorem 2 to apply we require that the coefficient of $\dfrac{d^2y}{dx^2}$ never takes the value zero. But $x^2 = 0$ when $x = 0$, and in this instance $x = 0$ has not been excluded from the domain.

However we can say that

$$y = Ax + Bx^2 \qquad (x > 0)$$

is the general solution of

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \qquad (x > 0).$$

Solutions to the exercises in Section 2

1. (i) The associated homogeneous equation is

$$\frac{d^2y}{dx^2} + 4y = 0.$$

Solving this gives the complementary function

$$y = A\cos 2x + B\sin 2x.$$

(ii) Trying a solution of the form $y = C$ (C constant) in $\dfrac{d^2y}{dx^2} + 4y = 8$ gives $0 + 4C = 8$, that is $C = 2$. Thus a particular solution is

$$y = 2.$$

(iii) By Theorem 1 the general solution (of the inhomogeneous equation) is

$$y = A\cos 2x + B\sin 2x + 2.$$

2. (i) For $y = mx + n$ to be a particular solution, we require

$$-2m + 2(mx + n) = 2x + 3 \qquad (\text{for all } x).$$

This can only be true for all values of x if m and n satisfy the simultaneous equations

$$\begin{aligned} -2m + 2n &= 3 \qquad (\text{equating constant terms}) \\ 2m &= 2 \qquad (\text{equating coefficients of } x). \end{aligned}$$

Solving for m and n we obtain

$$m = 1,\ n = \frac{5}{2}.$$

So a particular solution is

$$y = x + \frac{5}{2}.$$

(ii) For $y = mx + n$ to be a particular solution, we require

$$2m + (mx + n) = 2x.$$

This can only be true for all values of x if m and n satisfy the simultaneous equations

$$\begin{aligned} 2m + n &= 0 \qquad (\text{equating constant terms}) \\ m &= 2 \qquad (\text{equating coefficients of } x). \end{aligned}$$

Substituting $m = 2$ into $2m + n = 0$ gives $n = -4$, so a particular solution is

$$y = 2x - 4$$

3. If we try

$$y = me^{-x},$$

then $\dfrac{dy}{dx} = -me^{-x}$ and $\dfrac{d^2y}{dx^2} = me^{-x}$. So we need

$$2e^{-x} = 2me^{-x} + 2me^{-x} + me^{-x}.$$

This is satisfied if $2 = 5m$, that is $m = \dfrac{2}{5}$. So a particular solution is

$$y = \frac{2}{5}e^{-x}.$$

4. We can either try a sinusoidal solution directly or use the phasor method.

Method 1: We try

$$y = m\cos 3t + n\sin 3t,$$

then

$$\frac{dy}{dt} = 3n\cos 3t - 3m\sin 3t,$$

and

$$\frac{d^2y}{dt^2} = -9m\cos 3t - 9n\sin 3t$$

So we require

$$(-9m + 9n)\cos 3t + (-9n - 9m)\sin 3t = 9\cos 3t.$$

This can only be true for *all* values of t if m and n satisfy the simultaneous equations

$$-9m + 9n = 9$$
$$-9m - 9n = 0.$$

Solving these equations gives $m = -\frac{1}{2}$, $n = \frac{1}{2}$. So the particular solution is

$$y = -\tfrac{1}{2}\cos 3t + \tfrac{1}{2}\sin 3t.$$

Method 2: We rewrite the right-hand side of the differential equation as $\mathrm{Re}(9e^{3it})$ and try $y = \mathrm{Re}(ze^{3it})$. Using Procedure 2.2(a) we obtain

$$(-9 + 9i)z = 9,$$

so $z = \dfrac{9}{-9+9i} = \dfrac{1}{-1+i} = -\dfrac{1}{2} - \dfrac{1}{2}i$. The particular solution is therefore

$$y = -\tfrac{1}{2}\cos 3t + \tfrac{1}{2}\sin 3t.$$

5. *Method 1*: We try

$$y = m\cos 2t + n\sin 2t.$$

Then

$$\frac{dy}{dt} = 2n\cos 2t - 2m\sin 2t,$$

and

$$\frac{d^2y}{dt^2} = -4m\cos 2t - 4n\sin 2t.$$

So we need

$$-8m\cos 2t - 8n\sin 2t = 8\cos 2t + 16\sin 2t.$$

This can only be true for *all* values of t if $m = -1$ and $n = -2$. Thus a particular solution is

$$y = -\cos 2t - 2\sin 2t.$$

Method 2: We write the right-hand side of the differential equation as $\mathrm{Re}((8 - 16i)e^{2it})$ and try $y = \mathrm{Re}(ze^{2it})$. Using Procedure 2.2(a) we obtain

$$(-4 - 4)z = (8 - 16i).$$

So $z = -1 + 2i$ and the particular solution is

$$y = -\cos 2t - 2\sin 2t.$$

6.(i) We can see that $y = \cos 2t$ is a solution of $\dfrac{d^2y}{dt^2} + 4y = 0$, so we have an exceptional case. Two alternative methods of solution are given below.

Method 1: We try

$$y = t(m\cos 2t + n\sin 2t)$$

then

$$\frac{dy}{dt} = (m\cos 2t + n\sin 2t) + t(2n\cos 2t - 2m\sin 2t)$$

and

$$\frac{d^2y}{dt^2} = (4n\cos 2t - 4m\sin 2t) + t(-4m\cos 2t - 4n\sin 2t).$$

Substituting into $\dfrac{d^2y}{dt^2} + 4y$ gives $4n\cos 2t - 4m\sin 2t$, so we require

$$4n\cos 2t - 4m\sin 2t = 2\cos 2t.$$

This can only be satisfied for all values of t if $m = 0$ and $n = \frac{1}{2}$. So the particular solution is

$$y = \tfrac{1}{2}t\sin 2t.$$

Method 2: We rewrite the right-hand side of the equation as $\mathrm{Re}(2e^{2it})$ and try

$$y = \mathrm{Re}(tze^{2it})$$

then

$$\frac{dy}{dt} = \mathrm{Re}((1 + 2it)ze^{2it})$$

and

$$\frac{d^2y}{dt^2} = \mathrm{Re}((-4t + 4i)ze^{2it}).$$

Substituting into $\dfrac{d^2y}{dt^2} + 4y$ gives $\mathrm{Re}(4ize^{2it})$, so we need

$$4iz = 2.$$

Thus $z = -\frac{1}{2}i$ and the particular solution is

$$y = \tfrac{1}{2}t\sin 2t.$$

(ii) $\lambda = 1$ is a solution of the auxiliary equation $\lambda^2 - 3\lambda + 2 = 0$. So $y = e^x$ is a solution of the associated homogeneous equation, and we have an exceptional case. Following Procedure 2.3 we try

$$y = mxe^x.$$

Then $\dfrac{dy}{dx} = m(1 + x)e^x$ and $\dfrac{d^2y}{dx^2} = m(2 + x)e^x$ and so

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = ((2 + x) - 3(1 + x) + 2x)me^x = -me^x.$$

Thus we have a particular solution if $m = -4$. The particular solution is therefore

$$y = -4xe^x.$$

7. In this case the general solution of the associated homogeneous equation is $y = Ae^x + Bxe^x$. So not only is $y = me^x$ going to reduce to zero if substituted, but so is $y = mxe^x$. So we need to try something else to find a particular solution. The trick is to multiply by x again. That is, we try

$$y = mx^2e^x$$

Then $\dfrac{dy}{dx} = (2x + x^2)me^x$ and $\dfrac{d^2y}{dx^2} = (2x + x^2 + 2 + 2x)me^x$.

So

$$\begin{aligned}\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y &= (x^2 + 4x + 2 - 2(x^2 + 2x) + x^2)me^x \\ &= 2me^x.\end{aligned}$$

Thus we have a particular solution if $m = \frac{1}{2}$. The particular solution is therefore

$$y = \tfrac{1}{2}x^2e^x$$

8.(i) We know from Theorem 2 that we can deal with the terms on the right-hand side of the differential equation separately, and we know from Exercise 7 that $y = \frac{1}{2}x^2e^x$ is a particular solution for the case where the right-hand side is e^x.

Let us now deal with the e^{2x} term. To find a particular solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2x}, \qquad (1)$$

we can try $y = me^{2x}$. This satisfies the equation if

$$(4m - 4m + m)e^{2x} = e^{2x}.$$

That is, if $m = 1$. So a particular solution of Equation (1) is

$$y = e^{2x}$$

Using Theorem 2 a particular solution of the given differential equation is, therefore,

$$y = 4(\tfrac{1}{2}x^2e^x) - 3e^{2x} = 2x^2e^x - 3e^{2x}.$$

(ii) We can find a solution by trying

$$y = l + m\cos 4t + n\sin 4t$$

(or by the complex number method).

This is a solution if (substituting in the differential equation)

$$4(-16m\cos 4t - 16n\sin 4t) + 9(-4m\sin 4t + 4n\cos 4t) + 100(l + m\cos 4t + n\sin 4t) = 9\cos 4t + 100.$$

That is, if

$$100l + (36m + 36n)\cos 4t + (-36m + 36n)\sin 4t = 9\cos 4t + 100.$$

This can only be true for all values of t if

$$36m + 36n = 9 \quad \text{(equating coefficients of cos)}$$
$$-36m + 36n = 0 \quad \text{(equating coefficients of sin)}$$
$$100l = 100 \quad \text{(equating constant terms).}$$

Solving these simultaneous equations gives $l = 1$, $n = \frac{1}{8}$ and $m = \frac{1}{8}$. So a particular solution is

$$y = 1 + \tfrac{1}{8}(\cos 4t + \sin 4t).$$

9. (i) First find the complementary function, by solving

$$\frac{d^2\theta}{dt^2} + 4\theta = 0.$$

This has auxiliary equation

$$\lambda^2 + 4 = 0$$

which has solutions $\lambda = \pm 2i$. Hence the complementary function is

$$\theta = A\cos 2t + B\sin 2t$$

where A and B are arbitrary constants.

To find a particular solution, try

$$\theta = mt + n$$

This satisfies the given inhomogeneous differential equation if

$$4(mt + n) = 2t \quad \text{(for all } t\text{);}$$

that is, if $m = \frac{1}{2}$ and $n = 0$.

So the general solution of the given equation is

$$\theta = A\cos 2t + B\sin 2t + \tfrac{1}{2}t.$$

(ii) The associated homogeneous equation is

$$3\frac{d^2Y}{dx^2} - 2\frac{dY}{dx} - Y = 0.$$

Its auxiliary equation is

$$3\lambda^2 - 2\lambda - 1 = 0$$

which has solutions $\lambda = 1$ and $\lambda = -\frac{1}{3}$. Hence the complementary function is

$$Y = Ae^x + Be^{-\frac{1}{3}x},$$

where A and B are arbitrary constants.

To find a particular solution, try

$$Y = me^{2x} + n.$$

This satisfies the given equation if

$$3(4me^{2x}) - 2(2me^{2x}) - (me^{2x} + n) = e^{2x} + 3.$$

That is, if $m = \frac{1}{7}$ and $n = -3$.

So the general solution of the given equation is

$$Y = Ae^x + Be^{-\frac{1}{3}x} + \tfrac{1}{7}e^{2x} - 3$$

(iii) The auxiliary equation of the associated homogeneous equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

which has solutions $\lambda = 1$ and $\lambda = 2$. So the complementary function is

$$y = Ae^x + Be^{2x},$$

where A and B are arbitrary constants. Note that in finding a particular solution we have an exceptional case. Both of the functions on the right of the given differential equation are solutions of the associated homogeneous equation, so for a particular solution we should try

$$y = mxe^x + nxe^{2x}.$$

Then

$$\frac{dy}{dx} = (1 + x)me^x + (1 + 2x)ne^{2x}$$

and

$$\frac{d^2y}{dx^2} = (2 + x)me^x + (4 + 4x)ne^{2x}.$$

Substituting into $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$ gives $-me^x + ne^{2x}$ so we require

$$-me^x + ne^{2x} = 2e^x - 5e^{2x}.$$

This can only be true for all values of x if $m = -2$ and $n = -5$. Thus the general solution of the given differential equation is

$$y = Ae^x + Be^{2x} - 2xe^x - 5xe^{2x}.$$

(iv) The auxiliary equation of the associated homogeneous equation is

$$\lambda^2 + 2\omega\lambda + 2\omega^2 = 0.$$

Thus

$$\lambda = \tfrac{1}{2}(-2\omega \pm \sqrt{4\omega^2 - 8\omega^2}) = \omega(-1 \pm i).$$

So the complementary function is

$$y = e^{-\omega t}(A\cos \omega t + B\sin \omega t).$$

I will deal with the two parts of the right-hand side of the differential equation separately. (This is not necessary — I just prefer this way here.) A solution of

$$\frac{d^2y}{dt^2} + 2\omega\frac{dy}{dt} + 2\omega^2 y = \omega^2 h$$

can be found by putting $y = m$ (where m is a constant), which yields the particular solution

$$y = \tfrac{1}{2}h.$$

A solution of

$$\frac{d^2y}{dt^2} + 2\omega\frac{dy}{dt} + 2\omega^2 y = 5\sin 2\omega t. \qquad (1)$$

can be found by putting

$$y = m\cos 2\omega t + n\sin 2\omega t$$

(or by complex numbers). Then

$$\frac{dy}{dt} = 2\omega n\cos 2\omega t - 2\omega m\sin 2\omega t$$

and

$$\frac{d^2y}{dt^2} = -4\omega^2 m\cos 2\omega t - 4\omega^2 n\sin 2\omega t.$$

Substituting into the left-hand side of the differential equation gives

$$(-2m + 4n)\omega^2 \cos 2\omega t + (-4m - 2n)\omega^2 \sin 2\omega t.$$

We require this to equal $5 \sin 2\omega t$, which means that m and n must satisfy the simultaneous equations

$$-2m + 4n = 0$$
$$-4m - 2n = 5/\omega^2.$$

Solving these equations gives $m = -\dfrac{1}{\omega^2}$ and $n = -\dfrac{1}{2\omega^2}$. So a particular solution of Equation (1) is

$$y = -\frac{1}{2\omega^2}(2\cos 2\omega t + \sin 2\omega t).$$

Using Theorem 2, a particular solution of the given differential equation is

$$y = -\frac{1}{2\omega^2}(2\cos 2\omega t + \sin 2\omega t) + \tfrac{1}{2}h.$$

Hence the general solution is

$$y = e^{-\omega t}(A\cos \omega t + B\sin \omega t) - \frac{1}{2\omega^2}(2\cos 2\omega t + \sin 2\omega t) + \tfrac{1}{2}h.$$

10 (i) $y = me^{3x}$.

(ii) $y = m\cos 3x + n\sin 3x$ (or $y = \mathrm{Re}(ze^{3ix})$).

(iii) Note that the complementary function is $y = Ae^{2x} + Be^{-2x}$, so we need to try $y = mxe^{-2x}$ for a particular solution.

(iv) Note that the complementary function is $y = A\cos 2x + B\sin 2x$, so we must try

$$y = kx\sin 2x + lx\cos 2x + mx + n$$

for a particular solution (or $y = \mathrm{Re}(zxe^{2ix}) + mx + n$).

11 (i) The complementary function is

$$y = e^{-x}(A\cos x + B\sin x).$$

A particular solution is

$$y = 2.$$

So the general solution is

$$y = e^{-x}(A\cos x + B\sin x) + 2.$$

(ii) The complementary function is

$$y = Ae^{2x} + Be^{-2x}.$$

For a particular solution, we try $y = \mathrm{Re}(ze^{3ix})$. By Procedure 2.2(a)

$$(-9 - 4)z = -i$$

i.e. $z = \dfrac{1}{13}i$

Thus a particular solution is

$$y = -\frac{1}{13}\sin 3x.$$

Hence the general solution of the given differential equation is

$$y = Ae^{2x} + Be^{-2x} - \frac{1}{13}\sin 3x.$$

Solutions to the exercises in Section 3

1. (i) Do not be thrown by the different notation I have used here. The question could equally well be written:

$$\frac{d^2y}{dt^2} + 4y = 0; \quad y = 0 \text{ and } \frac{dy}{dt} = 1 \text{ when } t = \frac{\pi}{2},$$

where $y = u(t)$. (Letters other than y and t can be used for the variables, of course.) The auxiliary equation is

$$\lambda^2 + 4 = 0.$$

Therefore $\lambda = \pm 2i$ and the general solution is

$$u(t) = A\cos 2t + B\sin 2t.$$

Now $u(\pi/2) = 0$ and so

$$0 = A\sin \pi + B\cos \pi$$
$$= A \times 0 + B \times -1.$$

Therefore $B = 0$ and hence

$$u(t) = A\sin 2t.$$

Now $u'(t) = 2A\cos 2t$ and so the condition $u'(\pi/2) = 1$ gives

$$1 = 2A\cos \pi = -2A.$$

i.e. $A = -\frac{1}{2}$.

The required particular solution is therefore

$$u(t) = -\tfrac{1}{2}\sin 2t.$$

(ii) The complementary function is

$$y = Ae^x + Be^{2x}$$

(see Frame 1 for details).

A particular solution is

$$y = 2$$

The general solution is, therefore

$$y = Ae^x + Be^{2x} + 2.$$

When $x = 0$, $y = 4$ and so $4 = A + B + 2$. That is,

$$A + B = 2. \qquad (1)$$

Now $\dfrac{dy}{dx} = Ae^x + 2Be^{2x}$ and when $x = 0$, $\dfrac{dy}{dx} = -1$. Thus

$$-1 = A + 2B. \qquad (2)$$

Solving the Simultaneous Equations (1) and (2) gives $A = 5$, $B = -3$. The required particular solution is therefore

$$y = 5e^x - 3e^{2x} + 2.$$

2. This is covered in Frame 7 of the audio-tape.

3. Problems (i) and (iii) are initial condition problems, (ii) and (iv) are boundary condition problems.

In each case, the differential equation is the same. Its general solution is

$$u(x) = A\cos 2x + B\sin 2x.$$

The derivative is

$$u'(x) = -2A\sin 2x + 2B\cos 2x.$$

The solutions satisfying the given conditions are found below.

(i) The condition $u(0) = 1$ gives $A = 1$. The condition $u'(0) = 0$ gives $0 = 2B$. That is $B = 0$. The required solution is therefore

$$u(x) = \cos 2x,$$

(ii) The condition $u(0) = 0$ gives

$$0 = A + 0.$$

The condition $u(\pi/2) = 0$ gives

$$0 = A - 0.$$

Thus B is arbitrary and $A = 0$. Any solution of the form

$$u(x) = B\sin 2x$$

satisfies the problem.

(iii) The condition $u(0) = 0$ gives $A = 0$. The condition $u'(0) = 0$ gives $B = 0$. So the required solution is

$$u(x) = 0.$$

(iv) The condition $u(-\pi) = 1$ gives $A = 1$. The condition $u(\pi/4) = 2$ gives $B = 2$. So the required solution is

$$u(x) = \cos 2x + 2\sin 2x.$$

4. (i) The required solution is

$$y = 0$$

as we have a homogeneous differential equation with initial conditions of the form $y = 0$ and $\dfrac{dy}{dx} = 0$.

(ii) The complementary function is

$$y = Ae^{x} + Be^{-x}.$$

A particular solution is

$$y = -8.$$

So the general solution is

$$y = Ae^{x} + Be^{-x} - 8.$$

The condition '$y = 0$ at $x = 0$' gives

$$A + B - 8 = 0. \qquad (1)$$

Now $\dfrac{dy}{dx} = Ae^{x} - Be^{-x}$, and so the condition '$\dfrac{dy}{dx} = 0$ at $x = 0$' gives

$$A - B = 0. \qquad (2)$$

Solving the Simultaneous Equations (1) and (2) gives $A = B = 4$. Therefore, the required particular solution is

$$y = 4e^{x} + 4e^{-x} - 8.$$

(iii) The general solution is

$$x = (At + B)e^{-\omega t}$$

(see Exercise 2(i) of Section 1). The condition '$x = a$ at $t = 0$' gives

$$a = B.$$

Now $\dfrac{dx}{dt} = -\omega(At + B)e^{-\omega t} + Ae^{-\omega t}$, and so the condition '$\dfrac{dx}{dt} = 0$ at $t = 0$' gives

$$0 = -\omega B + A.$$

Thus $A = \omega B = a\omega$. The required particular solution is therefore

$$x = a(\omega t + 1)e^{-\omega t}.$$

5. (i) For first-order equations the theorem is as follows. Let

$$p(x)\frac{dy}{dx} + q(x)y = f(x)$$

be any linear first-order equation for which $p(x)$ does not take the value zero for any value of x. Then there is one and only one solution of this differential equation which also satisfies a condition of the form $y = b$ when $x = a$.

We can find some soluutions to the differential equation in (ii) by the following incautious manipulation. Separation of variables gives:

$$\int\frac{dy}{y} = \int\frac{dx}{x}$$

so $\quad \log_e y = \log_e x + C$

i.e. $\quad y = Ax.$

I say incautious because the manipulation performed is invalid if x or y is zero or negative. However it happens that the solutions found here are alright for all values of x and y, as I can show by checking.

Check: If

$$y = Ax$$

then

$$\frac{dy}{dx} = A$$

So

$$x\frac{dy}{dx} = Ax = y$$

for all real x, as required. Furthermore, if $x = 0$, then $y = 0$ (whatever A is). Hence *all* functions of the form

$$y = Ax$$

satisfy the given differential equation and initial condition.

(iii) There is not a *unique* solution in this case. We do have a linear equation here. What is wrong is that the condition concerning $p(x)$ is *not* satisfied (for $p(x) = x$ is zero when $x = 0$). This example illustrates the fact that the condition $p(x) \neq 0$ in the theorem is crucial. Without it, the theorem does not work.

6.(i) The general solution is

$$u(t) = e^{-2t}(A\cos t + B\sin t).$$

The condition $u(0) = 0$ gives $0 = A$ and hence

$$u(t) = Be^{-2t}\sin t$$

Thus

$$u'(t) = Be^{-2t}(\cos t - 2\sin t),$$

and so the condition $u'(0) = 2$ gives $2 = B$. The required solution is therefore

$$u(t) = 2e^{-2t}\sin t.$$

This is an initial condition problem.

(ii) The general solution is

$$u(t) = A\cos 3t + B\sin 3t.$$

The condition $u(0) = 0$ gives $0 = A$ and hence

$$u(t) = B\sin 3t.$$

Thus

$$u'(t) = 3B\cos 3t$$

and so the condition $u'(\pi/3) = 1$ gives $1 = 3B\cos\pi = -3B$. That is $B = -\frac{1}{3}$. The required solution is therefore

$$u(t) = -\frac{1}{3}\sin 3t.$$

This is a boundary condition problem.

(iii) We have a homogeneous equation, with initial conditions equal to zero. The required solution is therefore

$$y = 0.$$

This is an initial condition problem.

Solutions to the exercises in Section 4

1. The phasor of $\cos 5t$ is 1 and that of $\sin 5t$ is $-i$, thus the required phasor is

$$z = 4 + 3i$$

The amplitude is

$$|z| = \sqrt{4^2 + 3^2} = 5.$$

The phase is

$$\operatorname{Arg} z = \arctan\left(\tfrac{3}{4}\right) = 0.64.$$

2. The auxiliary equation is

$$\lambda^2 + 2\alpha\lambda + \omega^2 = 0$$

which has roots

$$\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$

The roots will be real provided $\alpha^2 - \omega^2 > 0$.

3. The graph is a sinusoid curve of angular frequency ω (the amplitude and phase depend on the initial conditions).

4. (i) We require $\alpha^2 - \omega^2$ to be negative and α to be positive (see table on page 36). Thus we require

$$0 < \alpha < \omega.$$

(ii) (a) For this equation $\alpha = 1.5$ and $\omega = 3$. Thus in this case $0 < \alpha < \omega$, and so the solution is a decaying oscillation. To ensure that the solution also fits the given initial conditions it must take the form shown in Figure 1 (starting at $y = 4$ with zero gradient).

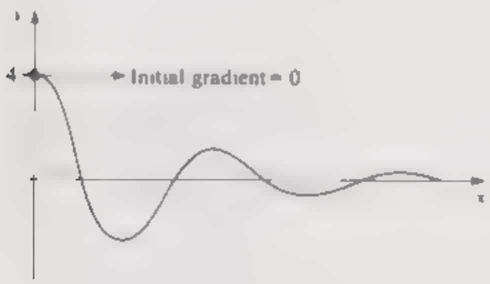


Figure 1

(b) For this equation $\alpha = 4$ and $\omega = 2$. Thus in this case $\alpha > \omega > 0$ and so the solution is a combination of decreasing exponentials. To satisfy the given initial conditions the solution must take the form shown in Figure 2.

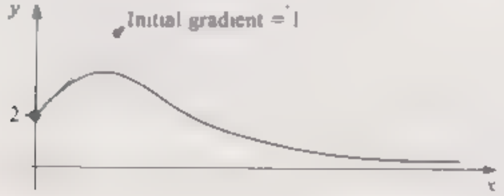


Figure 2

5. The solution is the sum of (I) a particular solution, which will be a sinusoid, and (II) the general solution of

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0.$$

Since $\alpha > 0$, this general solution will be decreasing, and will tend to zero as x becomes large and so is a transient. Since $\alpha < \omega$ this transient is oscillatory. The initial conditions only affect the transient term in the solution. So whatever the initial conditions the long-term behaviour of the solution is the same. It is a sinusoidal oscillation, of frequency ν. The amplitude and phase of this oscillation depend on ν, α and ω. (Example 1 on page 39 illustrates how the amplitude and phase can be calculated.)

6. The general solution of the differential equation is

$$y = A\cos 3x + B\sin 3x.$$

The angular frequency of this oscillation is 3. For the condition '$y = 4$ when $x = 0$' to be satisfied we require $A = 4$. Now $\dfrac{dy}{dx} = -3A\sin 3x + 3B\cos 3x$ and so for the condition '$\dfrac{dy}{dx} = 9$ when $x = 0$' to be satisfied we require $3B = 9$, that is $B = 3$. Thus the required particular solution is

$$y = 4\cos 3x + 3\sin 3x.$$

The phasor (see *Unit 5*) of this oscillation is

$$z = 4 - 3i$$

Hence the amplitude of the oscillation is

$$|z| = \sqrt{4^2 + 3^2} = 5,$$

and the phase is

$$\operatorname{Arg} z = -\arctan\left(\tfrac{3}{4}\right) = -0.64.$$

7. All the solutions of $u''(x) + \omega^2 u(x) = 0$ are sinusoidal oscillations of angular frequency ω. These solutions repeat themselves after a period $\dfrac{2\pi}{\omega}$ and so any solution $u(x)$ of this differential equation satisfies the condition

$$u(x) = u\left(x + \frac{2\pi}{\omega}\right)$$

for all values of x. In particular, when $x = a$ we have

$$u(a) = u\left(a + \frac{2\pi}{\omega}\right).$$

A solution can therefore never be found satisfying boundary conditions of the form

$$u(a) = k \quad \text{and} \quad u\left(a + \frac{2\pi}{\omega}\right) = l$$

if $k \neq l$. On the other hand, if $k = l$, then any solutions satisfying $u(a) = k$ will also satisfy $u\left(a + \dfrac{2\pi}{\omega}\right) = l \ (= k)$.

Consequently the two conditions are equivalent in this case, and so will only place one condition on the two arbitrary constants in the general solution. So for $k = l$, the solution is not unique.

8. From Example 7 of Section 1, the general solution is

$$x = Ae^{(-2+\sqrt{3})\omega t} + Be^{(-2-\sqrt{3})\omega t}.$$

For this to satisfy the initial condition '$x = a$ at $t = 0$' we require

$$a = A + B. \tag{1}$$

Now

$$\frac{dx}{dt} = (-2+\sqrt{3})\omega A e^{(-2+\sqrt{3})\omega t} - (2+\sqrt{3})\omega B e^{(-2-\sqrt{3})\omega t}$$

For this to satisfy the initial condition '$\frac{dx}{dt} = 0$ at $t = 0$' we require

$$0 = (\sqrt{3}-2)\omega A - (2+\sqrt{3})\omega B. \qquad (2)$$

Solving (1) and (2) gives

$$(\sqrt{3}-2)A = (2+\sqrt{3})B$$
$$= (2+\sqrt{3})(a-A).$$

Thus

$$A = \frac{(2+\sqrt{3})a}{2\sqrt{3}} = \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)a,$$

and

$$B = a - \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)a = \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)a.$$

So the required solution is

$$x = a\left(\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)e^{(-2+\sqrt{3})\omega t} + \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)e^{(-2-\sqrt{3})\omega t}\right)$$

9. The solutions of the auxiliary equation

$$\lambda^2 + 2\beta\omega\lambda + \omega^2 = 0$$

are

$$\lambda = \frac{-2\beta\omega \pm \sqrt{4\beta^2\omega^2 - 4\omega^2}}{2}$$
$$= \omega(-\beta \pm \sqrt{\beta^2 - 1}).$$

Hence (since $\beta^2 < 1$) the general solution is

$$x = e^{-\beta\omega t}\left(A\cos(\omega\sqrt{1-\beta^2})t + B\sin(\omega\sqrt{1-\beta^2})t\right)$$

For this to satisfy the conditions '$x = a$ at $t = 0$' we require

$$A = a.$$

Now

$$\frac{dx}{dt} = -\beta\omega e^{-\beta\omega t} \times$$
$$(A\cos(\omega\sqrt{1-\beta^2})t + B\sin(\omega\sqrt{1-\beta^2})t)$$
$$+ \omega\sqrt{1-\beta^2}\, e^{-\beta\omega t} \times$$
$$(-A\sin(\omega\sqrt{1-\beta^2})t + B\cos(\omega\sqrt{1-\beta^2})t).$$

For this to satisfy the condition '$\frac{dx}{dt} = 0$ at $t = 0$' we require

$$\omega(-A\beta + B\sqrt{1-\beta^2}) = 0$$

i.e.

$$B = \frac{A\beta}{\sqrt{1-\beta^2}} = \frac{a\beta}{\sqrt{1-\beta^2}}$$

So the required particular solution is

$$x = ae^{-\beta\omega t}\left(\cos(\omega\sqrt{1-\beta^2})t + \frac{\beta}{\sqrt{1-\beta^2}}\sin(\omega\sqrt{1-\beta^2})t\right).$$

This is a decaying oscillation, as sketched in Figure 3.

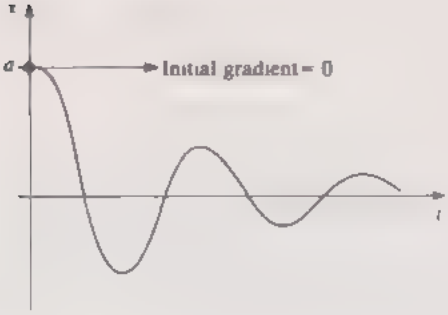


Figure 3

10. To find the phasor z of the steady-state solution, we try $y = \mathrm{Re}(ze^{5it})$. This gives

$$(4(5i)^2 + 9(5i) + 100)z = 9$$

i.e.

$$9(5i)z = 9.$$

So

$$z = \frac{1}{5i} = -\frac{1}{5}i.$$

Thus the amplitude of the steady-state solution is

$$|z| = \frac{1}{5},$$

and the phase is

$$\mathrm{Arg}\, z = -\frac{\pi}{2}.$$

Its angular frequency is 5. (The actual steady-state solution is, therefore, $x = \frac{1}{5}\cos\left(5t - \frac{\pi}{2}\right)$.)

11. If we try $y = \mathrm{Re}(ze^{ivt})$ we obtain the following equation for the phasor z of the steady-state solution:

$$(-v^2 + 2\beta\omega(iv) + \omega^2)z = 1$$

i.e.

$$z = \frac{1}{(\omega^2 - v^2) + (2\beta\omega v)i}.$$

The amplitude of the steady-state solution is

$$|z| = \frac{1}{\sqrt{(\omega^2 - v^2)^2 + (2\beta\omega v)^2}}.$$

12. The nature of the solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \omega^2 y = 0$$

depends on the nature of the roots of its auxiliary equation,

$$\lambda^2 + 2\alpha\lambda + \omega^2 = 0.$$

These are

$$\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$

They are real if $\alpha^2 - \omega^2$ is positive, complex if $\alpha^2 - \omega^2$ is negative. Let us look at the various cases in turn.

(i) α positive, $\alpha^2 - \omega^2$ positive

The roots are real, and both are negative. (The root $-\alpha + \sqrt{\alpha^2 - \omega^2}$ must be negative because $\alpha > \sqrt{\alpha^2 - \omega^2}$).

Thus the solution of the differential equation is a linear combination of $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ with λ_1 and λ_2 negative. Thus the typical solution in this case is a decreasing exponential.

(II) α negative, $\alpha^2 - \omega^2$ positive
Reasoning similar to that in Case *(I)* shows that in this case the solution is a linear combination of $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ with λ_1 and λ_2 positive. Thus the typical solution of the differential equation in this case is an increasing exponential.

(III) α positive, $\alpha^2 - \omega^2$ negative
In this case the roots of the auxiliary equation are complex:

$$\lambda = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}.$$

The solutions of the differential equation are linear combinations of $e^{-\alpha x}\cos(\sqrt{\omega^2 - \alpha^2}\,x)$ and $e^{-\alpha x}\sin(\sqrt{\omega^2 - \alpha^2}\,x)$. The typical solution is decreasing and oscillatory.

(IV) α negative, $\alpha^2 - \omega^2$ negative
The formulae for the solutions are as in Case *(III)*, but now $e^{-\alpha x}$ is an increasing exponential. Thus the solutions of the differential equation are oscillations of increasing amplitude.

Solutions to the exercises in Section 5

1. (i)
$$\frac{dy}{dx} = z$$
$$\frac{dz}{dx} = -4z - 5y + \cos x$$

(ii)
$$\frac{dy}{dt} = z$$
$$\frac{dz}{dt} = -32\sin y$$

2. In each case, replace $\frac{dy}{dx}$ by $(y_{r+1} - y_r)/h$, $\frac{dz}{dx}$ by $(z_{r+1} - z_r)/h$, and x, y and z by x_r, y_r and z_r.

(i) For these equations we obtain

$$\frac{y_{r+1} - y_r}{h} = z_r$$

and

$$\frac{z_{r+1} - z_r}{h} = -4z_r - 5y_r + \cos x_r,$$

which can be rearranged to give the pair of recurrence relations

$$y_{r+1} = y_r + hz_r$$
$$z_{r+1} = -5hy_r + (1 - 4h)z_r + h\cos x_r.$$

(ii) In this case we obtain

$$\frac{y_{r+1} - y_r}{h} = z_r$$

and

$$\frac{z_{r+1} - z_r}{h} = -32\sin y_r,$$

which can be rearranged to give the recurrence relations

$$y_{r+1} = y_r + hz_r$$
$$z_{r+1} = z_r - 32h\sin y_r.$$

3. (i)

x_r	y_r	z_r	$y_r + 0.2z_r$	$-0.2y_r + z_r$
0	0	1.0	0.2	1.0
0.2	0.2	1.0	0.4	0.96
0.4	0.4	0.96	0.592	0.88
0.6	0.592	0.88	0.768	0.762
0.8	0.768	0.762	0.920	0.608
1.0	0.920	0.608	1.042	0.424
1.2	1.042	0.424	1.127	0.216
1.4	1.127	0.216	1.170	−0.009
1.6	1.170	−0.009	1.168	0.243
1.8	1.168	0.243	1.119	−0.477
2.0	1.119	−0.477	1.024	0.701
2.2	1.024	−0.701	0.884	0.906
2.4	0.884	0.906	0.703	1.083
2.6	0.703	1.083	0.486	−1.224
2.8	0.486	1.224	0.241	1.321
3.0	0.241	−1.321	−0.023	1.369
3.2	**0.023**			

The Euler approximation to $y(3.2)$ is shown in bold type.

(ii) The general solution of $\frac{d^2y}{dx^2} + y = 0$ is

$$y = A\cos x + B\sin x.$$

The condition '$y = 0$ when $x = 0$' gives $A = 0$. The condition '$\frac{dy}{dx} = 1$ when $x = 0$' gives $B = 1$. So the required particular solution is:

$$y = \sin x.$$

(iii) The results are shown graphically in Figure 4. The general shape of the results is correct—a half cycle of an oscillation. However the results are not precisely accurate. (The values of y should not rise above 1). This inaccuracy is hardly surprising with so long a step-length as $h = 0.2$.

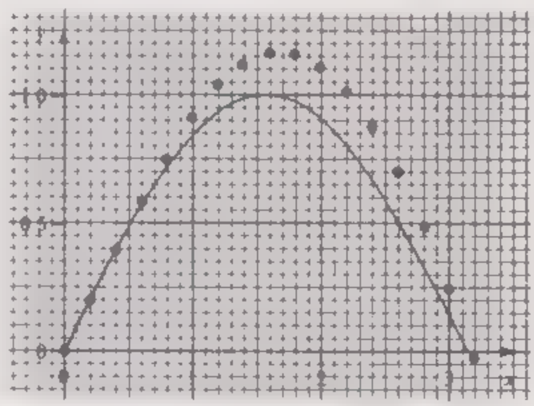


Figure 4

4.

x_r	y_r	z_r	$y_r + 0.2z_r$	$-4y_r - 1.4z_r$
0	1.0	1.0	1.2	−5.4
0.2	1.2	5.4	0.12	2.76
0.4	0.12	2.76	0.672	−4.344
0.6	0.672	−4.344	−0.197	3.394
0.8	−0.197	3.394	0.482	−3.964
1.0	**0.482**			

The Euler approximation to $y(1)$ is shown in bold type. The true solution is given by

$$y = \frac{1}{8}(11e^{-2x} - 3e^{-10x}).$$

Figure 5 shows a comparison of this true solution with the Euler approximations.

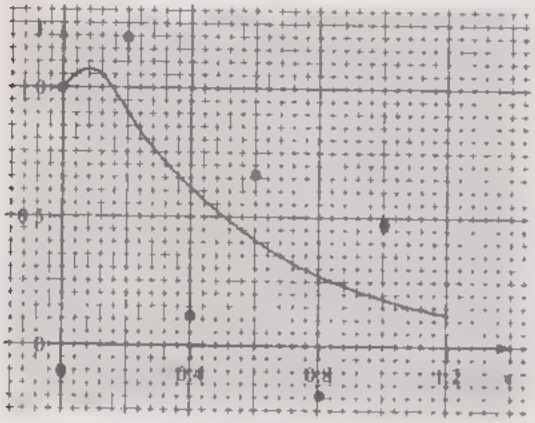


Figure 5

The method has failed miserably to give results that are anywhere near reasonable. However, we need not despair since if we use a small enough step-length, we can get useful results. Clearly, we have to be careful when we use numerical methods.

5.

x_r	y_r	z_r	$y_r + 0.2z_r$	$1.6z_r - 0.4y_r + 0.4$
0	1.0	1.0	1.2	1.6
0.2	1.2	1.6	1.52	2.48
0.4	1.52	2.48	2.016	3.76
0.6	2.016	3.76	2.768	5.610
0.8	2.768	5.610	3.890	
1.0	**3.890**			

The Euler approximation to $y(1)$ is shown in bold type. The true solution is given by

$$y = e^{2x} - e^{x} + 1.$$

The graph indicates that the numerical results have the same shape as the solution curve.

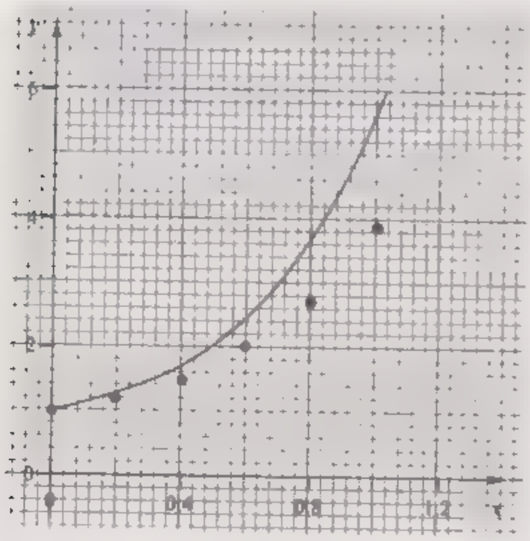


Figure 6

Solutions to the exercises in Section 6

1. (i) The auxiliary equation

$$\lambda^2 + 6\lambda + 5 = 0$$

has roots $\lambda = -5$, $\lambda = -1$ and so the general solution is

$$y = Ae^{-5x} + Be^{-x}.$$

The initial condition '$y = 2$ at $x = 0$' is satisfied if

$$2 = A + B. \tag{1}$$

The condition '$\frac{dy}{dx} = 1$ at $x = 0$' is satisfied if

$$1 = -5A - B. \tag{2}$$

Solving Equations (1) and (2) gives $A = -\frac{3}{4}$, $B = \frac{11}{4}$. The required particular solution is therefore

$$y = \frac{1}{4}(11e^{-x} - 3e^{-5x}).$$

(ii) The complementary function is

$$y = Ae^{-2x} + B.$$

A particular solution is $y = me^{2x}$, where

$$4me^{2x} + 4me^{2x} = e^{2x}$$

i.e. $m = \frac{1}{8}$. The general solution is therefore

$$y = Ae^{-2x} + B + \frac{1}{8}e^{2x}$$

The initial conditions '$y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$' give

$$0 = A + B + \frac{1}{8},$$

$$1 = -2A + \frac{1}{4}.$$

Solving these simultaneous equations we obtain $A = -\frac{3}{8}$, $B = \frac{1}{4}$. The required particular solution is therefore

$$y = \frac{1}{8}e^{2x} + \frac{1}{4} - \frac{3}{8}e^{-2x}$$

(iii) The complementary function is

$$y = Ae^{3t} + Be^{-3t}.$$

A particular solution of

$$\frac{d^2y}{dt^2} - 9y = 18$$

is

$$y = -2$$

A particular solution of

$$\frac{d^2y}{dt^2} - 9y = 3\sin 3t$$

is a sinusoid with phasor z, where (putting $y = \mathrm{Re}(ze^{3it})$)

$$-18z = -3i,$$

i.e. $z = \frac{1}{6}i$.

That is

$$y = -\frac{1}{6}\sin 3t.$$

The general solution is therefore

$$y = Ae^{3t} + B^{-3t} - 2 - \frac{1}{6}\sin 3t.$$

The conditions '$y = 0$ and $\frac{dy}{dt} = 0$ at $t = 0$' give

$$0 = A + B - 2$$

$$0 = 3A - 3B - \frac{1}{2}.$$

Solving these simultaneous equations gives $A = \frac{13}{12}$, $B = \frac{11}{12}$.

The required particular solution is therefore

$$y = \frac{1}{12}(13e^{3t} + 11e^{-3t} - 24 - 2\sin 3t).$$

(iv) This can be solved by direct integration. We have

$$\frac{d\theta}{du} = \int \frac{du}{u^2} = -\frac{1}{u} + A.$$

We require $\frac{d\theta}{du} = 2$ when $u = 1$, so $A = 3$. That is

$$\frac{d\theta}{du} = 3 - \frac{1}{u}.$$

Hence

$$\theta = \int \left(3 - \frac{1}{u}\right) du$$
$$= 3u - \log_e u + B.$$

The condition '$\theta = 1$ when $u = 1$' gives

$$1 = 3 + B.$$

Thus $B = -2$ and the required particular solution is

$$\theta = 3u - \log_e u - 2.$$

2. (i) The complementary function is

$$y = A\cos 4t + B\sin 4t.$$

This is an exceptional case. We find a particular solution (of the form $y = t \times$ (a sinusoid)) by putting $y = \text{Re}(zte^{4it})$. Then

$$\frac{dy}{dt} = ze^{4it}(1 + 4it),$$

and

$$\frac{d^2y}{dt^2} = ze^{4it}(4i + 4i(1 + 4it)) = ze^{4it}(8i - 16t).$$

So

$$\frac{d^2y}{dt^2} + 16y = 8ize^{4it}.$$

Thus the phasor z satisfies

$$8iz = 4 - 8i,$$

i.e. $z = -1 - \frac{1}{2}i$. So a particular solution is

$$y = t(-\cos 4t + \frac{1}{2}\sin 4t).$$

Hence the general solution is

$$y = A\cos 4t + B\sin 4t - t\cos 4t + \frac{1}{2}t\sin 4t.$$

(ii) The complementary function is

$$y = Ae^{-2t} + Bte^{-2t}.$$

To find a particular solution, try

$$y = lt^2e^{-2t} + mt + n.$$

Then

$$\frac{dy}{dt} = l(2t - 2t^2)e^{-2t} + m$$

and

$$\frac{d^2y}{dt^2} = l(2 - 4t - 4t + 4t^2)e^{-2t} = l(2 - 8t + 4t^2)e^{-2t}.$$

Thus

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dx} + 4y = le^{-2t}\left((2 - 8t + 4t^2) + 4(2t - 2t^2) + 4t^2\right)$$
$$+ 4m + 4(mt + n)$$
$$= 2le^{-2t} + 4mt + 4(m + n).$$

So we require

$$2le^{-2t} + 4mt + 4(m + n) = 3e^{-2t} + 8t.$$

Therefore $2l = 3$, $\quad 4m = 8$, $\quad 4(m + n) = 0$. That is

$$l = \frac{3}{2}, \qquad m = 2, \qquad n = -2.$$

The general solution is therefore

$$y = Ae^{-2t} + Bte^{-2t} + \frac{3}{2}t^2e^{-2t} + 2t - 2.$$

3. (i) Since $(6^2 - 4 \times 1 \times 1) > 0$ and $6 > 0$ the graph has the form

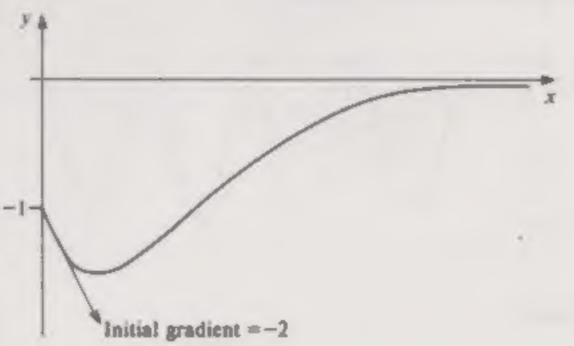


Figure 7

(ii) Since $(1^2 - 4 \times 6 \times 1) < 0$ and $1 > 0$ the graph has the form

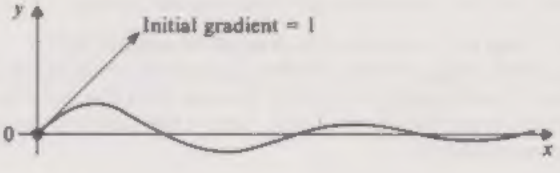


Figure 8

4. (i) Since the coefficient of $\frac{dx}{dt}$ is positive the complementary function is a transient. The long-term motion is therefore independent of the initial conditions. The non-transient term will be a particular solution of the form

$$x = A\cos(3t + \phi) + m$$

where A, ϕ and m are constants. It is easiest to deal with the constant term m separately. For $x = m$ to be a solution of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = 2$$

we require $4m = 2$. That is $m = \frac{1}{2}$ and the solution is

$$x = \frac{1}{2}.$$

To find the sinusoidal solution of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = 2\sin 3t,$$

we can use phasors. The phasor z of this solution is found by putting $x = \mathrm{Re}(ze^{3it})$ (remembering that the phasor of $2\sin 3t$ is $-2i$). So

$$((3i)^2 + 3i + 4)z = -2i$$

i.e. $z = \dfrac{-2i}{-5+3i} = \dfrac{2i(5+3i)}{34} = \dfrac{2}{34}(-3+5i)$.

Thus A (the amplitude) is given by

$$A = |z| = \frac{2}{\sqrt{34}}$$

and ϕ (the phase) is given by

$$\phi = \mathrm{Arg}\, z = \pi - \arctan\frac{5}{3} = 2.11.$$

Thus the steady-state solution is

$$x = \frac{1}{2} + \frac{2}{\sqrt{34}}\cos(3t + 2.11).$$

The graph is shown in Figure 9.

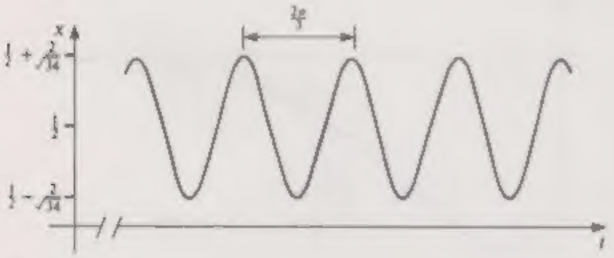


Figure 9

The smallest value taken by x during this part of the motion is

$$x = \frac{1}{2} - \frac{2}{\sqrt{34}}$$

This is positive, so x does not take negative values.

(ii) 'a long time after the object is set in motion' means 'sufficiently long after the motion has started for the transient term to have become negligible'. We can spell this out more precisely if we find the transient. This is the complementary function, which is

$$x = e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right).$$

The term that determines how rapidly this transient dies away is clearly, $e^{-\frac{1}{2}t}$. Precisely when it is reasonable to neglect this term depends on the particular problem under consideration. However, it would often be adequate to require that $e^{-\frac{1}{2}t} < 10^{-4}$. This occurs when $t > 2\log_e 10^4 \simeq 18.4$.

5. The general solution of the differential equation is

$$u(x) = A\cos\lambda\pi x + B\sin\lambda\pi x.$$

For $u(0) = 0$ we need

$$0 = A.$$

For $u(1) = 0$ we then need

$$0 = B\sin\lambda\pi.$$

For a non-zero solution we require that B is not zero, so we need

$$\sin\lambda\pi = 0.$$

This occurs if λ is an integer.

6. (i) This is a consequence of Theorem 1 of Section 3. The differential equation $u''(x) + u(x) = 0$ is of the form specified in that theorem, so there is exactly one solution of this equation satisfying a given set of initial conditions. Hence there is one and only one function S satisfying the given definition, and so this is an adequate definition. A similar comment applies to the definition of C.

(ii) (a) Since S is a solution of Equation (1):

$$S'' + S = 0.$$

Differentiating this gives

$$S''' + S' = 0.$$

i.e. $(S')'' + S' = 0.$

So S' is a solution of Equation (1).

(b) Also

$S'(0) = 1$ (by the definition of S)

and

$S''(0) = -S(0)$ (from Equation (1))
$= 0$ (by the definition of S).

So the function S' is a solution of Equation (1), that satisfies the initial conditions

$$S'(0) = 1, \quad (S')'(0) = 0.$$

But the function C satisfies the same initial conditions, and by Theorem 1 of Section 3 there is only *one* solution of Equation (1) satisfying these initial conditions. Hence

$$C = S'.$$

(iii) C' is a solution of Equation (1), by an argument just like that in part (ii). Also

$C'(0) = 0,$ (by the definition of C)

and

$C''(0) = -C(0)$ (from Equation (1))
$= -1$ (by the definition of C)

Now the function $-S$ is also a solution of Equation (1), and satisfies the initial conditions

$$-S(0) = 0, \quad -S'(0) = -1.$$

Hence, by Theorem 1 of Section 3:

$$C' = -S.$$

(iv) The derivative of $C^2 + S^2$ is

$$2CC' + 2SS' = 2C(-S) + 2SC = 0.$$

Hence $C^2 + S^2$ must be a constant function. So

$$C^2 + S^2 = C^2(0) + S^2(0) = 1.$$

The Open University
0 335 14035 1